PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 2, October 1983

MAPPING THEOREMS ON MESOCOMPACT SPACES

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ABSTRACT. In this paper we prove two mapping theorems on mesocompact spaces: (1) Perfect mappings preserve mesocompactness; (2) Closed mappings preserve normal mesocompactness.

The main results of this paper are two mapping theorems on mesocompact spaces. Mesocompactness was defined in J. R. Boone [**4**] and studied by V. J. Mancuso [**10**] and J. R. Boone [**4, 5**]. Mancuso [**10**] intended to prove that perfect mappings preserve mesocompactness, but his proof was incorrect. J. R. Boone [**5**] noticed the error in Mancuso's proof but he gave a proof only for a special case (the domains of the mappings were assumed to be normal). Our Theorem 1 solves the Mancuso problem. Boone [**6**] studied k-quotient mappings and proved that k-quotient, closed mappings preserve normal mesocompactness. Our Theorem 2 improves the foregoing result by omitting the condition "k-quotient" in the statement.

In this paper, normal spaces are assumed to be T_1, and all mappings are continuous and surjective. Let $\mathcal{U}$ be a collection of subsets of X, the union $\bigcup\{U\colon U \in \mathcal{U}\}$ is denoted by $\mathcal{U}^*$. For any $B \subset X$, let $(\mathcal{U})_B = \{U \in \mathcal{U}\colon U \cap B \neq \varnothing\}$ and $(\mathcal{U})_{\{x\}}$ is replaced by $(\mathcal{U})_x$. For the meanings of concepts used without definition in this paper, see [**8** and **9**].

DEFINITION 1. A collection $\mathcal{U}$ of subsets of X is called compact-finite, if for each compact subset $K \subset X$, $(\mathcal{U})_K$ is finite.

DEFINITION 2 [**4**]. A topological space X is called mesocompact if every open cover of the space has a compact-finite open refinement.

It is well known that

$$\text{paracompact} \to \text{mesocompact} \to \text{metacompact}$$

and none of the implications can be reversed.

DEFINITION 3. Let $\mathcal{U}$ and $\mathcal{V}$ be two collections of subsets of X, we say that $\mathcal{U}$ is a compactwise W-refinement of $\mathcal{V}$, if $\mathcal{U}^* = \mathcal{V}^*$ and for each compact subset $K \subset X$, the collection $(\mathcal{U})_K$ is a partial refinement of some finite subcollection $\mathcal{V}'$ of $\mathcal{V}$.

In the proofs of the following lemmas and Proposition 1, we use the techniques invented by Junnila [**8** and **9**].

Received by the editors September 13, 1982 and, in revised form, January 19, 1983.

1980 *Mathematics Subject Classification*. Primary 54D20, 54C10.

Key words and phrases. Paracompact, metacompact, mesocompact, isocompact, semiopen cover, directed cover, compact-covering mapping, pseudo-open mapping.

0002-9939/83 $1.00 + $.25 per page

LEMMA 1. *Let $\{\mathcal{U}_n\}_{n\in N}$ be a sequence of open covers of X such that for each $n \in N$, $\mathcal{U}_{n+1}$ is a compactwise W-refinement of $\mathcal{U}_n$. Then $\mathcal{U}_1$ has an open refinement $\mathcal{V} = \bigcup_{n=2}^{\infty}\mathcal{V}_n$ such that each $\mathcal{V}_n$ is a compact-finite collection.*

PROOF. Because each $\mathcal{U}_{n+1}$ is also a pointwise W-refinement of $\mathcal{U}_n$, by [**8**, Proposition 2.2], $\mathcal{U}_1$ has an open refinement $\mathcal{V} = \bigcup_{n=2}^{\infty}\mathcal{V}_n$, such that for each $n \in N$ and $B \subset X$, if $(\mathcal{U}_{n+1})_B$ is a partial refinement of a subcollection $\mathcal{U}'$ of $\mathcal{U}_n$, then $|(\mathcal{V}_{n+1})_B| \leqslant |\mathcal{U}'|$. Now for each compact subset $K \subset X$, there exists a finite subcollection $\mathcal{U}'$ of $\mathcal{U}_n$ such that $(\mathcal{U}_{n+1})_K$ is a partial refinement of $\mathcal{U}'$, so $|(\mathcal{V}_{n+1})_K| \leqslant |\mathcal{U}'| < \infty$, that is, $\mathcal{V}_n$ is compact-finite for each $n \geqslant 2$.

LEMMA 2. *If an open cover of a topological space has a compact-finite semiopen refinement, then the cover has an open compactwise W-refinement.*

PROOF. The proof is similar to the proof of [**9**, Lemma 1.2].

PROPOSITION 1. *The following conditions are mutually equivalent for a topological space*:

(1) *The space is mesocompact.*

(2) *Every open cover of the space has a compact-finite semiopen refinement.*

(3) *Every open cover of the space has an open compactwise W-refinement.*

(4) *Every directed open cover of the space has a closure-preserving closed refinement which is refined by the collection consisting of all compact subsets.*

PROOF. (1) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (3) follows from Lemma 2.

(3) $\Rightarrow$ (1). By Lemma 1, every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{W} = \bigcup_{n=1}^{\infty}\mathcal{W}_n$, each $\mathcal{W}_n$ is compact-finite. For each $n \in N$, let $R_n = \bigcup_{k=1}^{n}\mathcal{W}_k^*$, then $\mathcal{R} = \{R_n\colon n \in N\}$ is a directed open cover of X. Let $\mathcal{P}$ be an open compactwise W-refinement of $\mathcal{R}$. Let $F_0 = \varnothing$ and for each $n \in N$, let $F_n = \{x \in X\colon \mathrm{St}(x, \mathcal{P}) \subset R_n\}$, note that each set F_n is closed. For each $n \in N$, let $\mathcal{V}_n = \{W - F_{n-1}\colon W \in \mathcal{W}_n\}$, it is easily seen that $\mathcal{V} = \bigcup_{n=1}^{\infty}\mathcal{V}_n$ is an open refinement of $\mathcal{U}$ [**8**, Theorem 3.1, proof (ii) $\Rightarrow$ (i)].

Let $V \subset X$ be any compact subset. Because $\mathcal{R}$ is directed, there exists an integer $n(K)$ such that $\mathrm{St}(K, \mathcal{P}) \subset R_{n(K)}$, that is $K \subset F_{n(K)}$. Then we have

$$|(\mathcal{V})_K| \leqslant \sum_{n=1}^{\infty} |(\mathcal{V}_n)_K| = \sum_{n=1}^{n(K)} |(\mathcal{V}_n)_K| \leqslant \sum_{n=1}^{n(K)} |(\mathcal{W}_n)_K| < \infty.$$

Therefore $\mathcal{U}$ is compact-finite.

(1) $\Rightarrow$ (4). Let $\mathcal{U}$ be any directed open cover of X, then $\mathcal{U}$ has a compact-finite open refinement $\mathcal{V}$. For each $U \in \mathcal{U}$, let $F(U) = \{x \in X\colon \mathrm{St}(x, \mathcal{V}) \subset U\}$, then $\mathcal{F} = \{F(U)\colon U \in \mathcal{U}\}$ is a closure-preserving closed refinement of $\mathcal{U}$ [**8**, Lemma 2.3, proof (ii) $\Rightarrow$ (i)]. Suppose $K \subset X$ is a compact subset. Since $\mathcal{U}$ is directed, there exists some $U \in \mathcal{U}$, such that $\mathrm{St}(K, \mathcal{V}) \subset U$, that is $K \subset F(U)$. Therefore $\mathcal{F}$ is refined by the collection consisting of all compact subsets.

(4) $\Rightarrow$ (3). By [**8**, Theorem 3.1], X is metacompact. Suppose $\mathcal{U}$ is any open cover of X, then $\mathcal{U}$ has a point-finite open refinement $\mathcal{V}$. Let $\tilde{\mathcal{V}}$ be the collection consisting of

all finite unions of sets from $\mathcal{V}$. As a directed open cover of X, $\tilde{\mathcal{V}}$ has a closure-preserving closed refinement $\mathcal{F}$ which is refined by the collection consisting of all compact subsets. For each $x \in X$, let $W(x) = \bigcap(\mathcal{V})_x - \bigcup\{F \in \mathcal{F}: x \notin F\}$. $\mathcal{W} = \{W(x): x \in X\}$ is an open cover of X. For each $F \in \mathcal{F}$, let $\mathcal{V}_F$ be the finite subcollection of $\mathcal{V}$ such that $F \subset \mathcal{V}_F^* \in \tilde{\mathcal{V}}$. Then $(\mathcal{W})_F$ is a partial refinement of $\mathcal{V}_F$ [**8**, Lemma 2.3, Proof (i) $\Rightarrow$ (iii)].

Now for each compact subset $K \subset X$, there exists an $F(K) \in \mathcal{F}$ such that $K \subset F(K)$, so $(\mathcal{W})_K$ is a partial refinement of a finite subcollection $\mathcal{V}_{F(K)}$ of $\mathcal{V}$, that is, $\mathcal{W}$ is an open compactwise W-refinement of $\mathcal{V}$ as well as of $\mathcal{U}$.

Definition 4 (Michael [**12**]). A mapping f: $X \to Y$ is called compact-covering if, whenever K is a compact set in Y, there exists a compact set C in X such that $f(C) = K$.

Proposition 2. *The image of a mesocompact space under a closed and compact-covering mapping is mesocompact.*

Proof. Let $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ be a directed open cover of Y, then $\mathcal{U} = \{f^{-1}(V_\beta)\}_{\beta \in B}$ is a directed open cover of X. By Proposition 1(4), $\mathcal{U}$ has a closure-preserving closed refinement $\mathcal{F} = \{F_\alpha\}_{\alpha \in A}$, which is refined by the collection consisting of all compact subsets of X. Since f is closed and compact-covering, $\{f(F_\alpha)\}_{\alpha \in A}$ is a closure-preserving closed refinement of $\mathcal{V}$, and is refined by the collection consisting of all compact subsets of Y. By Proposition 1(4), Y is mesocompact.

Since every perfect mapping is compact-covering (e.g. [**13**]), we obtain the following theorem.

Theorem 1. *The image of a mesocompact space under a perfect mapping is mesocompact.*

Definition 5 (Bacon [**3**]). A topological space is called isocompact if every countably compact closed subset of X is compact.

E. Michael [**11**] proved that if X is a paracompact space and f: $X \to Y$ a closed mapping, then f is also a compact-covering mapping. In his proof, the paracompactness is used only for turning a countably compact closed subset of a normal space to a compact one. We have the following lemma.

Lemma 3. *A closed mapping f from a normal isocompact space X onto a space Y is also a compact-covering mapping.*

Theorem 2. *The image of a normal mesocompact space under a closed mapping is normal mesocompact.*

Proof. Let f be a closed mapping from a normal mesocompact space X onto a space Y. It is well known that Y is normal. Because mesocompact space is isocompact (Arens-Dugundji [**1**]), so by Lemma 3, f is also a compact-covering mapping. By Proposition 2, Y is mesocompact.

Remark. Junnila [**9**] proved that the image of a paracompact space under a pseudo-open and compact mapping is metacompact in order to answer the question of Arhangel'skii [**2**] affirmatively. Junnila's proof for the paracompact case can easily be modified to cover the mesocompact case. On the other hand, the image of a metacompact space under an open and compact mapping is not necessarily metacompact (Chaber [7]).

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