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A THEOREM ON THE CARDINALITY OF κ-TOTAL SPACES

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ABSTRACT. Throughout this article, κ denotes an arbitrary infinite cardinal number. In 1979, A. A. Gryzlov strengthened a well-known result of A. V. Arhangel'skii by proving that every compact T_1-space of pseudocharacter κ has cardinality $\leqslant 2^\kappa$. Using techniques similar to Gryzlov's, we prove that every 2^κ-total, T_1-space of pseudocharacter $\leqslant \kappa$ is compact and hence of cardinality $\leqslant 2^\kappa$. Some related results and examples are given.

The terminology and notation we use are standard, except that κ and λ always denote cardinal numbers, and other Greek letters denote ordinal numbers. The adherence of a filter base $\mathfrak{F}$ on a topological space, $\bigcap\{\bar{F}: F \in \mathfrak{F}\}$ is denoted ad $\mathfrak{F}$.

Recall that a topological space X is called *κ-bounded* [**GFW**] (*κ-total* [**V**]) provided that for every subset Y of X, if $|Y| \leqslant \kappa$, then there is a compact subset K of X with $Y \subset K$ (then every filter base on Y has an adherent point in X). A space X is called *initially κ-compact* [**AU**, p. 20] if any of the following equivalent conditions hold: (i) for every open cover $\mathcal{V}$ of X, if $|\mathcal{V}| \leqslant \kappa$ then $\mathcal{V}$ has a finite subcover; (ii) for every filter base $\mathfrak{F}$ on X, if $|\mathfrak{F}| \leqslant \kappa$ then ad $\mathfrak{F} \neq \varnothing$; and (iii) for every infinite subset E of X, if $|E| \leqslant \kappa$ then some point $p \in X$ is a *complete accumulation point of E*, i.e., for every neighborhood V of p, $|V \cap E| = |E|$. It is known (e.g., see [**V**]) and easily shown that for a space X: (i) κ-bounded $\Rightarrow$ κ-total $\Rightarrow$ initially κ-compact; (ii) if $2^\kappa \leqslant \lambda$ then initially λ-compact $\Rightarrow$ κ-total; and (iii) if X is regular then κ-total $\Rightarrow$ κ-bounded.

Two other definitions we shall use are the following. A T_1-(Hausdorff) space X is said to have *pseudocharacter* (*H-pseudocharacter*) λ if λ is the smallest cardinal number having the following property: for each point $x \in X$ there exists a family $\mathfrak{B}$ of neighborhoods of x such that $|\mathfrak{B}| \leqslant \lambda$ and $\{x\} = \bigcap \mathfrak{B}$ ($\{x\} = \bigcap\{\bar{B}: B \in \mathfrak{B}\}$).

In order to obtain our extension of Gryzlov's theorem [**G**], two lemmas are needed. The first one was obtained in [**G**] under the assumption that $Y (= X)$ is compact.

LEMMA 1. *Let X be a T_1-space of pseudocharacter $\leqslant \kappa$ and suppose that Y is a subset of X such that every filter base on Y has an adherent point in X. Then every initially κ-compact subset Z of Y is compact.*

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PROOF. Let $\mathfrak{F}$ be a maximal filter base of closed subsets of Z. We prove $\bigcap \mathfrak{F} \neq \varnothing$. Choose an adherent point $x \in X$ of $\mathfrak{F}$, and let $\mathfrak{B}$ be a family of open neighborhoods of x in X such that $\bigcap \mathfrak{B} = \{x\}$ and $|\mathfrak{B}| \leqslant \kappa$. For each $B \in \mathfrak{B}$, it follows from the maximality of $\mathfrak{F}$ that there exists a set $F_B \in \mathfrak{F}$ with $F_B \subset B$. Then by the initial κ-compactness of Z, $\varnothing \neq \bigcap \{F_B\colon B \in \mathfrak{B}\}$, and since $\bigcap \{F_B\colon B \in \mathfrak{B}\} \subset Z \cap (\bigcap \mathfrak{B}) \subset Z \cap \{x\}$, one has $x \in Z$ and thus $x \in Z \cap \operatorname{ad} \mathfrak{F} = \bigcap \mathfrak{F}$.

The next lemma is due to V. Saks [**SS**, Theorem 4.5] for $\kappa \geqslant \omega_0$ and to Z. Frolík [**F**] for $\kappa = \omega_0$. For completeness, we include a proof outline.

LEMMA 2. *Let X be an initially κ-compact space and suppose $A \subset X$ has cardinality $\leqslant 2^\kappa$. Then there exists an initially κ-compact subset G of X such that $A \subset G$ and $|G| \leqslant 2^\kappa$.*

PROOF. For each infinite subset I of X with $|I| \leqslant \kappa$, choose one complete accumulation point p_I of I in X, and for each subset B of X, define $B' = B \cup \{p_I\colon I \subset B\}$. Next, define $G_0 = A$ and $G_\alpha = (\bigcup \{G_\beta\colon \beta < \alpha\})'$ for $0 < \alpha < \kappa^+$, and note that the set $G = \bigcup \{G_\alpha\colon \alpha < \kappa^+\}$ has the desired properties.

The proof in [**G**, Theorem 1] can now be given to establish the following

THEOREM 3. *If X is a 2^κ-total, T_1-space of pseudocharacter $\leqslant \kappa$, then X is compact and of cardinality $\leqslant 2^\kappa$.*

PROOF. For each point $x \in X$, let $\mathfrak{B}_x$ be a family of open neighborhoods of x with $|\mathfrak{B}_x| \leqslant \kappa$ and $\bigcap \mathfrak{B}_x = \{x\}$, and let c be a mapping into X such that $c(A) \in A$ for every nonempty subset A of X. For each nonempty subset S of X with $|S| \leqslant 2^\kappa$, use Lemmas 1 and 2 to select a compact subset S^* of X such that $|S^*| \leqslant 2^\kappa$ and S^* contains the set

$$S \cup \left\{c\left(X \setminus \bigcup \mathfrak{F}\right)\colon S \subset \bigcup \mathfrak{F},\ X \neq \bigcup \mathfrak{F},\ |\mathfrak{F}| < \omega_0, \text{ and } \mathfrak{F} \subset \bigcup \{\mathfrak{B}_x\colon x \in S\}\right\}.$$

Choose a point $x \in X$ and let $G_0 = \{x\}$. Define $G_\alpha = (\bigcup \{G_\beta\colon \beta < \alpha\})^*$ for $0 < \alpha < \kappa^+$, and let $G = \bigcup \{G_\alpha\colon \alpha < \kappa^+\}$. Then $|G| \leqslant 2^\kappa$ and G is initially κ-compact, so G is, in fact, compact. Moreover, $G = X$, for if $X \setminus G \neq \varnothing$, there must also exist $\mathfrak{F}$ with $G \subset \bigcup \mathfrak{F}$, $X \neq \bigcup \mathfrak{F}$, $|\mathfrak{F}| < \omega_0$, and $\mathfrak{F} \subset \bigcup \{\mathfrak{B}_x\colon x \in G\}$, but then it would be the case that for some $\alpha < \kappa^+$, $\mathfrak{F} \subset \bigcup \{\mathfrak{B}_x\colon x \in G_\alpha\}$ and hence one would have $c(X \setminus \bigcup \mathfrak{F}) \notin \bigcup \mathfrak{F}$, whereas $G \subset \bigcup \mathfrak{F}$ and $c(X \setminus \bigcup \mathfrak{F}) \in G_{\alpha+1} \subset G$.

In [**D**, Corollary 4.4], A. Dow proved that (CH) every first countable, initially ω_1-compact, Tychonoff space is compact. At the Special Session on Rings of Continuous Functions at the 1982 Annual Meeting of the American Mathematical Society, he proved that "Tychonoff" can be weakened to "Hausdorff" and noted that E. van Douwen independently had obtained the same result. Using Theorem 3, one can extend their result further. First we obtain a lemma.

LEMMA 4. *If a Hausdorff space X is initially λ-compact and of H-pseudocharacter $\leqslant \lambda$, then it is regular and of character $\leqslant \lambda$.*

PROOF. Consider any point $x \in X$ and family $\mathfrak{B}$ of neighborhoods of x such that $|\mathfrak{B}| \leqslant \lambda$ and $\{x\} = \bigcap \{\bar{B}\colon B \in \mathfrak{B}\}$. Let V be an open neighborhood of x. If $\bar{C} \cap (X \setminus V) \neq \varnothing$ for every $C \in \mathcal{C}$, the family of all finite intersections of members

of $\mathcal{B}$, then by the initial λ-compactness of $X \setminus V$, $(X \setminus V) \cap \bigcap \{\bar{C}\colon C \in C\} \neq \varnothing$. But the latter could not occur since $\bigcap \{\bar{C}\colon C \in \mathcal{C}\} = \{x\}$ and $x \in V$. Thus, for some $C \in \mathcal{C}$, $V \supset \bar{C}$.

REMARK. The result that every first countable, countably compact, Hausdorff space is regular is obtained in [**AU**, p. 28]. In [**A**], a proof is given that a regular G_δ-point in a countably compact space has a countable neighborhood base. The result that every initially κ-compact, Hausdorff space of character κ is regular is due to Saks [**S**, Theorem 5.7].

COROLLARY 5. *If X is an initially 2^κ-compact, Hausdorff space of pseudocharacter $\leqslant \kappa$ and H-pseudocharacter $\leqslant 2^\kappa$, then it is compact.*

PROOF. By Theorem 3, it suffices to prove that X is 2^κ-bounded. Applying Lemma 4 with $\lambda = 2^\kappa$, one sees that X is regular and thus of H-pseudocharacter $\leqslant \kappa$, and applying it again with $\lambda = \kappa$, one sees that X is of character $\leqslant \kappa$. Since X is also Hausdorff, it follows that if $A \subset X$ and $|A| \leqslant 2^\kappa$ then $|\bar{A}| \leqslant 2^\kappa$. Thus, whenever $A \subset X$ and $|A| \leqslant 2^\kappa$, $\bar{A}$ is both initially 2^κ-compact and of cardinality $\leqslant 2^\kappa$, i.e., $\bar{A}$ is compact.

An example shows that in Theorem 3, 2^κ-total cannot be weakened to κ-total. Indeed, for every cardinal number λ, there exists a κ-bounded, completely normal T_1-space X of character $\leqslant \kappa$ such that $|X| = \lambda$.

EXAMPLE 6. Let $X = \{\alpha\colon \alpha \leqslant \lambda \text{ and } \mathrm{cf}(\alpha) \leqslant \kappa\}$, with the subspace topology X inherits from $[0, \lambda]$, where the latter has the order topology.

PROOF. Since X contains all members of $[0, \lambda]$, except possibly certain limit ordinals, $|X| = \lambda$. To see that X has character $\leqslant \kappa$, note that if $\alpha \in X$ is a limit ordinal, there exists $B \subset [0, \alpha)$ with $|B| = \mathrm{cf}(\alpha)$ and $\sup B = \alpha$, so that $\{X \cap (\beta, \alpha]\colon \beta \in B\}$ is a neighborhood base for α in X of cardinality $\leqslant \kappa$. Other points of X are isolated.

To prove that X is κ-bounded, consider any subset A of X, and let $K = \mathrm{cl}_X A$. If K fails to be compact, then it cannot be a closed subset of $[0, \lambda]$, so there must exist $\beta \leqslant \lambda$ such that $\mathrm{cf}(\beta) > \kappa$ and $A \cap (\alpha, \beta] \neq \varnothing$ for all $\alpha < \beta$. But from the latter it would follow that $|A| \geqslant \mathrm{cf}(\beta) > \kappa$.

Note that the space X need not be even locally compact, for if there exist $\beta \in X$ and $B \subset [0, \beta) \cap ([0, \lambda] \setminus X)$ such that $\sup B = \beta$, then X fails to be locally compact at β.

Since examples abound of κ-bounded spaces of character $\leqslant \kappa$ that are not locally compact or of cardinality $\leqslant 2^\kappa$, it is obvious that in Theorem 3, "pseudocharacter $\leqslant \kappa$" cannot be weakened to "pseudocharacter $\leqslant 2^\kappa$." The following example shows, moreover, that *for every infinite cardinal number κ, there exists a κ-total, Hausdorff space of character κ^+ which is neither κ-bounded nor initially κ^+-compact.*

EXAMPLE 7. Suppose that B is any compact Hausdorff space of character κ^+ having a dense subset D and a κ-bounded but not initially κ^+-compact subset Z such that $|D| \leqslant \kappa$ and $D \cap Z = \varnothing$. Denote by X the Hausdorff space whose points are the same as those of B but whose topology is

$$\{S \cup (T \setminus Z)\colon S \text{ and } T \text{ are open subsets of } B\}.$$

Then X is a Hausdorff space of character κ^+ which is neither κ-bounded nor initially κ^+-compact, for D is a dense subset, $|D| \leqslant \kappa$, and $\overline{D} = X$ fails to be initially κ^+-compact (since X has a closed but not initially κ^+-compact subset, namely, Z). Furthermore, because Z is κ-bounded and B is compact, it is immediate that X is κ-total.

Next, to see that such spaces B can be found, take $B = \Pi\{X_\alpha: \alpha < \kappa^+\}$, where for each $\alpha < \kappa^+$, $X_\alpha = \{0, 1\}$, with the discrete topology. Let $Z = \{x \in B: |\{\alpha: x_\alpha \neq 0\}| \leqslant \kappa\}$. Then B is compact Hausdorff, and it is easy to show that Z is κ-bounded but not initially κ^+-compact. One can find an acceptable set D as follows. Choose (e.g., by viewing the indexing set κ^+ as being identified with a subset of the product space $\Pi\{X_\alpha: \alpha < 2^\kappa\}$) a family $\P$ of partitions of the set κ^+ such that: (i) $|\P| \leqslant \kappa$; (ii) $|\mathcal{P}| < \omega_0$ for each $\mathcal{P} \in \P$; and (iii) for each nonempty finite subset F of κ^+, there exists $\mathcal{P} \in \P$ such that every member of $\mathcal{P}$ contains exactly one element of F. Then one can take D to be $\{x \in B \setminus Z$: for some $\mathcal{P} \in \P$, $x|P$ is constant for each $P \in \mathcal{P}\}$.

We conclude by stating a question not settled above.

QUESTION. If κ is an infinite cardinal number, does there exist an initially 2^κ-compact, T_1-space of pseudocharacter $\leqslant \kappa$ which fails to be compact?

Note that by Corollary 5 any such space could not be regular, and if it were Hausdorff then it would also have to be κ-bounded, for any Hausdorff space having a dense subset of cardinality $\leqslant \kappa$ is of H-pseudocharacter $\leqslant 2^\kappa$.

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