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A COMPOSITION THEOREM FOR δ-CODES

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ABSTRACT. If Golay complementary sequences (or equivalently a two-symbol δ-code) of length n and a Turyn δ-code of length t exist then four-symbol δ-codes of length $(2n+1)t$ can be composed. Therefore new families of Hadamard matrices of orders $4uw$ and $20uw$ can be constructed, where $u = (2^{a+1}10^b26^c + 1)t$ for odd $t \leqslant 59$ or $t = 2^d10^e26^f + 1$ (all a, b, c, d, e, and $f \geqslant 0$), and w is the order of Williamson matrices.

The following result has been obtained: If Golay complementary sequences of length s and Turyn base sequences for length t (see [**1**]) exist, then four-symbol δ-codes of length $(2s+1)t$ can be composed. Consequently four-symbol δ-codes of length $u = (2^{a+1}10^b26^c + 1)t$ can be constructed for odd $t \leqslant 59$ or $t = 2^d10^e26^f + 1$, where a, b, c, d, e and f are nonnegative integers. Therefore new families of Hadamard matrices of orders $4uw$ and $20uw$ can be constructed, where w is the order of Williamson matrices.

In this paper, many notations and definitions are the same as in [**1**]. *Turyn base sequences for length* $t = 2m + p$ (abbreviated as $TBS(t)$) are four $(1,-1)$-sequences $(A, B; C, D)$, respectively, of lengths $m+p$ and m pairwise such that

$$(1) \qquad A = (a_k)_{m+p}, \quad B = (b_k)_{m+p}, \quad C = (c_k)_m \quad \text{and} \quad D = (d_k)_m$$

having zero auto-correlation sum, i.e. $a(j) + b(j) + c(j) + d(j) = 0$ for $j \neq 0$, where $p(j) = \sum_{k=1}^{n-j} p_k p_{k+j}$ for $P = (p_k)_n$. Another characterization of $TBS(t)$ is that the associated polynomials satisfy

$$(2) \qquad |A|^2 + |B|^2 + |C|^2 + |D|^2 = 2t, \quad \text{for any } z \in \mathbf{K} = \{z \in \mathbf{C} : |z| = 1\},$$

where we use the same notation P to represent a given sequence $(p_k)_n$ and its associated polynomial $P(z) = \sum_1^n p_k z^{k-1}$; $\mathbf{K}$ is the unit circle and $\mathbf{C}$ is the complex field.

Turyn sequences of length t (abbreviated as $TS(t)$) (or *a four-symbol δ-code of length t*) are four $(0, \pm 1)$-sequences (I, J, K, L) of length t, where $I = (i_h)_t$, $J = (j_h)_t$, $K = (k_h)_t$ and $L = (l_h)_t$, satisfying $i(h) + j(h) + k(h) + l(h) = 0$ for $h \neq 0$ and $|i_h| + |j_h| + |k_h| + |l_h| = 1$ for each h. It is known that the associated polynomials $TS(t)$ satisfy $|I|^2 + |J|^2 + |K|^2 + |L|^2 = t$ for any z in $\mathbf{K}$.

A regular δ-code of length u (abbreviated as $RD(u)$) is a quad (Q, R, S, T) of $(0, \pm 1)$-sequences of length u such that

$$(3) \qquad Q = I + J, \quad R = I - J, \quad S = K + L \quad \text{and} \quad T = K - L,$$

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which is derived from $TS(u)$: (I, J, K, L). The existence of $RD(u)$ and that of $TS(u)$ are equivalent since, from (3), we also have $I = (Q + R)/2$, $J = (Q - R)/2$, $K = (S + T)/2$ and $L = (S - T)/2$. We note here that the associated polynomials of $RD(u)$ satisfy

$$|Q|^2 + |R|^2 + |S|^2 + |T|^2 = 2u \quad \text{for any } z \text{ in } \mathbf{K}, \tag{4}$$

and

(5) either $q_k r_k = \pm 1$ and $s_k = t_k = 0$, or $q_k = r_k = 0$ and $s_k t_k = \pm 1$, for each k, where q_k, r_k, s_k, and t_k are the kth term of the sequences Q, R, S, and T, respectively.

We first prove the following *Lagrange identity theorem for polynomials*, which is the key to our constructions for $RD(u)$.

THEOREM 1. *Let A, B, C, D, E, F, G and H be polynomials in $z \in \mathbf{K}$ with real coefficients. Also let*

$$\begin{aligned} W &= -B'E + AF' + CG + DH, & Y &= -D'E - CF + AG' - BH, \\ X &= A'E + BF' + DG' - CH', & Z &= C'E - DF + BG + AH', \end{aligned} \tag{6}$$

where $P' = P(z^{-1})$ for $P = P(z)$. Then

$$\begin{aligned} &|W|^2 + |X|^2 + |Y|^2 + |Z|^2 \\ &\quad = \left(|A|^2 + |B|^2 + |C|^2 + |D|^2\right)\left(|E|^2 + |F|^2 + |G|^2 + |H|^2\right). \end{aligned} \tag{7}$$

PROOF. Since

$$|W|^2 + |X|^2 + |Y|^2 + |Z|^2 = WW' + XX' + YY' + ZZ' \quad \text{and} \tag{8}$$

$$\begin{aligned} W' &= -BE' + A'F + C'G' + D'H', \\ X' &= AE' + B'F + D'G - C'H, \\ Y' &= -DE' - C'F' + A'G - B'H', \\ Z' &= CE' - D'F' + B'G' + A'H, \end{aligned} \tag{9}$$

by substituting the right-hand sides of (6) and (9) into that of (8), and by expansions, simplifications and regrouping, (7) can be derived.

Golay complementary sequences of length s (abbreviated as $GCL(s)$) are two $(1, -1)$-sequences: $\mathbf{F} = (f_k)_s$ and $\mathbf{G} = (g_k)_s$ having zero auto-correlation sum, $f(j) + g(j) = 0$ for $j \neq 0$. Another characterization of $GCL(s)$ (see [**2**]) is that the associated polynomials satisfy $|\mathbf{F}|^2 + |\mathbf{G}|^2 = 2s$ for any z in $\mathbf{K}$. $GCL(s)$ are known to exist for $s = 2^a 10^b 26^c$, where a, b, and c are nonnegative integers (see [**3**]).

THEOREM 2. *Let $(A, B; C, D)$ be $TBS(t)$ of* (1) *and $\mathbf{F} = (f_k)_s$ and $\mathbf{G} = (g_k)_s$ be $GCL(s)$, where $s = 2r$. Then the following (Q, R, S, T) are the associated polynomials of $RD((2s + 1)t)$.*

$$\begin{aligned} Q &= (-B'E + AF' + CG)U, & S &= (-D'E - CF + AG')V, \\ R &= (A'E + BF' + DG')U, & T &= (C'E - DF + BG)V, \end{aligned} \tag{$*$}$$

where $P = P(z)$, *for* $P = A, B, C, D$, *and* $E = z^{(s+1)t-1}$,

$$F = \sum_{k=-r}^{r-1} f_{r+k+1} z^{(2k+1)t+M}, \qquad G = \sum_{k=-r}^{r-1} g_{r+k+1} z^{(2k+1)t};$$

$M = m + p = t - m$; $U = z^{(s-1)t+M}$, *and* $V = z^{st}$.

PROOF. By applying Theorem 1 to (Q, R, S, T) and observing that $UU' = 1 = VV'$, $|F|^2 + |G|^2 = 2s$, $|E|^2 = 1$ and $H = 0$, we obtain, from (2),

$$\begin{aligned} |Q|^2 + |R|^2 + |S|^2 + |T|^2 &= \left(|A|^2 + |B|^2 + |C|^2 + |D|^2\right)\left(|E|^2 + |F|^2 + |G|^2\right) \\ &= 2t(2s+1), \end{aligned}$$

for any z in $\mathbf{K}$. Therefore, (4) is satisfied. And since

$$F' = \sum_{k=-r}^{r-1} f_{r+k+1} z^{-(2k+1)t-M} = \sum_{h=-r}^{r-1} f_{r-h} z^{(2h+1)t-M}$$

and

$$G' = \sum_{k=-r}^{r-1} g_{r+k+1} z^{-(2k+1)t} = \sum_{h=-r}^{r-1} g_{r-h} z^{(2h+1)t},$$

we have from $(*)$

$$(**) \qquad Q = \sum_{h=-r}^{r-1} \left(A f_{r-h} z^{(2h+1)t} + C g_{r+h+1} z^{(2h+1)t+M}\right) z^{(s-1)t} - B' z^{2st+M-1},$$

$$R = \sum_{h=-r}^{r-1} \left(B f_{r-h} z^{(2h+1)t} + D g_{r-h} z^{(2h+1)t+M}\right) z^{(s-1)t} + A' z^{2st+M-1},$$

$$S = \sum_{h=-r}^{r-1} \left(A g_{r-h} z^{(2h+1)t} - C f_{r+h+1} z^{(2h+1)t+M}\right) z^{st} - D' z^{(2s+1)t-1},$$

$$T = \sum_{h=-r}^{r-1} \left(B g_{r+h+1} z^{(2h+1)t} - D f_{r+h+1} z^{(2h+1)t+M}\right) z^{st} + C' z^{(2s+1)t-1}.$$

Consequently, (5) is also satisfied. When $s = 1$, Theorem 2 takes the following form: i.e. $r = \frac{1}{2}$, thus $F = z^M$ and $G = 1$, if $f_1 = g_1 = 1$.

COROLLARY. *When* $s = 1$, *we let* $F = z^M$, $G = 1$, *and* E, U *and* V *be as in Theorem 2. Then* (Q, R, S, T) *is* $RD(3t)$.

For example, from the corollary, we obtain $RD(3t)$ as follows:

$$\begin{aligned} Q &= (A, C; 0, 0; -B', 0), & S &= (0, 0; A, -C; 0, -D'), \\ R &= (B, D; 0, 0; A', 0), & T &= (0, 0; B, -D; 0, C'). \end{aligned}$$

We note here that the above (Q, R, S, T) corresponds to $(f, g^*, e, -h^*)$ in the proof of the theorem of [**1**]. By setting $\mathbf{F} = (-1, 1)$ and $\mathbf{G} = (1, 1)$ as $GCL(2)$, we obtain the following $RD(5t)$ from $(**)$.

$$\begin{aligned} Q &= (A, C; 0, 0; -A, C; 0, 0; -B', 0), & S &= (0, 0; A, C; 0, 0; A, -C; 0, -D'), \\ R &= (B, D; 0, 0; -B, D; 0, 0; A', 0), & T &= (0, 0; B, D; 0, 0; B, -D; 0, C'). \end{aligned}$$

In general, we obtain the following $RD((2s+1)t)$ from Theorem 2, (**).

$$(Af_s, Cg_1; 0,0;\ AF_{s-1}, Cg_2; 0,0; \ldots;\ Af_1, Cg_s; 0,0; -B', 0),$$
$$(Bf_s, Dg_s; 0,0;\ Bf_{s-1}, Dg_{s-1}; 0,0; \ldots;\ Bf_1, Dg_1; 0,0;\ A', 0),$$
$$(0,0;\ Ag_s, -Cf_1; 0,0;\ Ag_{s-1}, -Cf_2; 0,0; \ldots; 0,0;\ Ag_1, -Cf_s; 0, -D'),$$
$$(0,0;\ Bg_1, -Df_1; 0,0;\ Bg_2, -Df_2; 0,0; \ldots; 0,0;\ Bg_s, -Df_s; 0, C').$$

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