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A SMOOTH SCISSORS CONGRUENCE PROBLEM

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ABSTRACT. Classifying space techniques are used to solve a smooth version of the classical scissors congruence problem.

1. Introduction.

1.1 *The classical problem* [**8**]. Let B be the abelian group generated by the set of polygons in the plane, modulo the subgroup generated by elements $P - \Sigma P_i$, where $P \amalg P_i$ is a subdivision of a polygon P. Any subgroup G of the group of affine motions of the plane acts on B. The problem is to compute the quotient group $H_0(G; B)$ of B by the subgroup generated by elements $gb - b$, with $g \in G$, $b \in B$.

1.2 *A smooth version*. Our purpose is to state and solve a smooth version of the problem. Instead of polygons transforming under affine maps, we consider smooth curves transforming under diffeomorphisms.

The basic tool is a space M (2.1) whose first singular integral homology group $H_1 M$ is a smooth version of the group B. Diffeomorphisms of the plane act on M and hence on $H_1 M$. We employ a slight modification of a standard spectral sequence in our calculations.

1.3 *Organization*. §2 states the key definitions and results; the major proof is in §3. §4 contains the proof of a lemma, and §5 discusses the spectral sequence.

I would like to thank the referee for suggestions and for a simplification in the proof of Lemma 3.5.

2. Results. We require some definitions.

2.1 DEFINITION. Let M be the one-manifold of C^∞ nonsingular curves in $\mathbf{R}^2$, defined as

$$M = \amalg (a, b)_f / \sim$$

where for each C^∞ nonsingular embedding f of an interval (a, b) to $\mathbf{R}^2$ we take a copy $(a, b)_f$ of (a, b), and where if $x \in (a, b)_f$ and $y \in (c, d)_g$ we set $x \sim y$ if and only if there exist neighborhoods U of x in $(a, b)_f$ and V of y in $(c, d)_g$ and a (not necessarily orientation preserving) diffeomorphism $h\colon U \to V$ such that $f|_U = g \circ h$.

M is a one-dimensional C^∞ nonorientable non-Hausdorff manifold; let $i\colon M \to \mathbf{R}^2$ denote the obvious immersion. If $g\colon U \to V$ is a diffeomorphism between open sets in $\mathbf{R}^2$, let $i^*g\colon i^{-1}U \to i^{-1}V$ denote the resulting diffeomorphism between the open

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subsets $i^{-1}U$ and $i^{-1}V$ of M. Let H_1M denote the first singular integral homology group of M.

2.2 DEFINITION. Let $H_0(\Gamma^\infty; H_1M)$ (resp. $H_0(\Gamma^\Omega; H_1M)$) denote the quotient group of H_1M by the subgroup generated by elements $(i^*g)_*b - b$, where $g: U \to V$ is an orientation preserving (resp. area and orientation preserving) C^∞ diffeomorphism between open subsets of $\mathbf{R}^2$, and $b \in H_1M$ has support in $i^{-1}U$.

Our problem is to compute the groups just defined.

2.3 EXAMPLE. The Figure 8 curve (with orientation given by the arrow in Figure 1) defines an element of H_1M. Here is one demonstration that this element is 0 in $H_0(\Gamma^\Omega; H_1M)$. The dotted curve indicates a part of M used in each step.

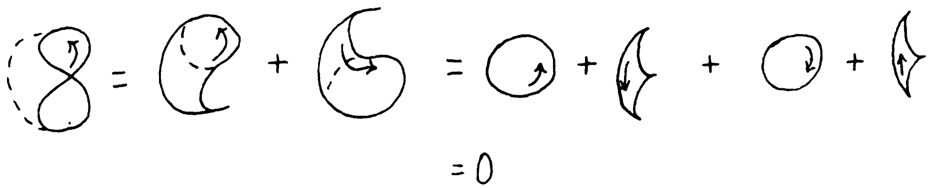

FIGURE 1

2.4 DEFINITION. (i) The *winding maps* W: $H_0(\Gamma^\infty; H_1M) \to \mathbf{Z}$, W: $H_0(\Gamma^\Omega; H_1M) \to \mathbf{Z}$. The tangent line field of M defines a map from M to $\mathbf{R}P^1$, and hence from H_1M to $H_1\mathbf{R}P^1$. Picking an isomorphism of $H_1\mathbf{R}P^1$ with $\mathbf{Z}$ gives a map $H_1M \to \mathbf{Z}$, which pushes down to the maps W.

(ii) The *area map* A: $H_0(\Gamma^\Omega; H_1M) \to \mathbf{R}$: If $b \in H_1M$, let $\underline{A}(b) = \int_{[b]} x\, dy$ (here $[b]$ denotes the one-current of $\mathbf{R}^2$ associated to b). $\underline{A}(b)$ is the "algebraic area enclosed by b". $\underline{A}$ pushes down to the map A.

2.5 THEOREM. *The maps* W: $H_0(\Gamma^\infty; H_1M) \to \mathbf{Z}$ *and* $W \oplus A$: $H_0(\Gamma^\Omega; H_1M) \to \mathbf{Z} \oplus \mathbf{R}$ *are isomorphisms.*

2.6 REMARK. We compare 2.5 with the classical result. Let B (as in 1.1) be the abelian group generated by polygons in the plane, modulo the subgroup generated by subdivisions. Let $AG1$ and $AS1$ denote the group of orientation preserving affine maps of the plane and the subgroup of area and orientation preserving maps, respectively. Then **[8]** $H_0(AG1; B) = 0$, and area gives an isomorphism A: $H_0(AS1; B) \to \mathbf{R}$. There is no "winding map".

2.7 REMARK. If in Definition 2.1 we glue the intervals $(a, b)_f$ together using *orientation preserving* diffeomorphisms h, we obtain a double cover $\tilde{M}$ of M, the *one-manifold of* C^∞ *oriented nonsingular curves in* $\mathbf{R}^2$. There are winding maps W: $H_0(\Gamma^\Omega; H_1\tilde{M}) \to \mathbf{Z}$ and W: $H_0(\Gamma^\infty; H_1\tilde{M}) \to \mathbf{Z}$ defined via the tangent unit vector map from M to S^1, and an area map A: $H_0(\Gamma^\Omega; H_1\tilde{M}) \to \mathbf{R}$. One can prove that W: $H_0(\Gamma^\infty; H_1\tilde{M}) \to \mathbf{Z}$ and $W \oplus A$: $H_0(\Gamma^\Omega; H_1\tilde{M}) \to \mathbf{Z} \oplus \mathbf{R}$ are isomorphisms.

3. Proof of 2.5. We shall prove that W: $H_0(\Gamma^\Omega; H_1M) \to \mathbf{Z} \oplus \mathbf{R}$ is an isomorphism. The proof for W: $H_0(\Gamma^\infty; H_1M) \to \mathbf{Z}$ is almost identical (see Remark 3.6).

Recall that a topological category is a small category whose sets of objects and morphisms are topologized such that the structure maps of the category are continuous. The nerve of a topological category is a simplicial space; we use Segal's "thick" realization (denoted $\|\cdot\|$ in [**9**, Appendix A]) to produce a classifying space functor $|\cdot|$ from topological categories to topological spaces.

3.1 DEFINITION. Let Γ^{Ω} be the topological category whose space of objects is $\mathbf{R}^2$, and whose space of morphisms, denoted Γ_1^{Ω}, is the space of germs of C^{∞} area and orientation preserving diffeomorphisms of $\mathbf{R}^2$, with the sheaf topology. Let D, R: $\Gamma_1^{\Omega} \to \mathbf{R}^2$ denote the domain and range maps of Γ^{Ω}.

The classifying space $|\Gamma^{\Omega}|$ is the "classifying space for C^{∞} codimension 2 foliation, with a transverse orientation and area form".

3.2 DEFINITION. Let $\Gamma^{\Omega}\backslash M$ be the topological category of the action Γ^{Ω} on M; the space of objects of $\Gamma^{\Omega}\backslash M$ is M, and the space of morphisms $(\Gamma^{\Omega}\backslash M)_1$ of $\Gamma^{\Omega}\backslash M$ is the pullback:

$$\begin{array}{ccc} (\Gamma^{\Omega}\backslash M)_1 & \to & \Gamma_1^{\Omega} \\ \downarrow D & & \downarrow D \\ M & \xrightarrow{i} & \mathbf{R}^2 \end{array}$$

Let i: $\Gamma^{\Omega}\backslash M \to \Gamma^{\Omega}$ denote the continuous functor covering the map i.

Now we claim [**2**]

3.3 PROPOSITION. *There is a first quadrant spectral sequence E^*_{pq}, with differential d^n of bidegree $(-n, n-1)$, which abuts to $H_{p+1}|\Gamma^{\Omega}\backslash M|$ and such that $E^2_{p0} = H_p|\Gamma^{\Omega}|$ and $E^2_{01} = H_0(\Gamma^{\Omega}; H_1 M)$.*

The spectral sequence is discussed in §5. To apply it to the proof of 2.5 we need two lemmas.

3.4 LEMMA [**4**, 2.6 AND **6**, LEMMA 1]. $H_1|\Gamma^{\Omega}| = 0$ *and* $H_2|\Gamma^{\Omega}| = \mathbf{Z} \oplus \mathbf{R}$.

3.5 LEMMA. $H_1|\Gamma^{\Omega}\backslash M| = \mathbf{Z}/2$.

The proof of 3.5 is in §4.

PROOF OF THEOREM 2.5. Let $K \oplus C$: $H_2|\Gamma^{\Omega}| \to \mathbf{Z} \oplus \mathbf{R}$ be the isomorphism of Lemma 3.4. Considering the spectral sequence 3.3, 2.5 will follow from the facts that $A \circ d^2 \circ C^{-1}$: $\mathbf{R} \to \mathbf{R}$ is an isomorphism and that the image of $W \circ d^2 \circ K^{-1}$ is $2\mathbf{Z}$ (here d^2 is the differential for the E^2-term). These facts will follow from an explicit description of d^2: $H_2|\Gamma^{\Omega}| \to H_0(\Gamma^{\Omega}; H_1 M)$ for elements of $H_2|\Gamma^{\Omega}|$ represented by closed oriented two-manifolds with an area form.

Let X be such a two-manifold, and let $[X] \in H_2|\Gamma^{\Omega}|$ be the corresponding homology class; $K[X]$ is the Euler characteristic of X, and $C[X]$ is the area of X. To describe $d^2[X]$, give a C^{∞} cell decomposition $X = \amalg \sigma_i$ of X as in Figure 2. Each cell σ_i can be mapped to $\mathbf{R}^2$ by an orientation and area preserving diffeomorphism f_i; the boundary of $f_i\sigma_i$, with orientation inherited from X, gives a cycle $[\partial f_i \sigma_i] \in H_1 M$. Then $d^2[X] = \Sigma[\partial f_i \sigma_i]$ is well defined in $H_0(\Gamma^{\Omega}; H_1 M)$ and independent of the choice of C^{∞} cell decomposition of X.

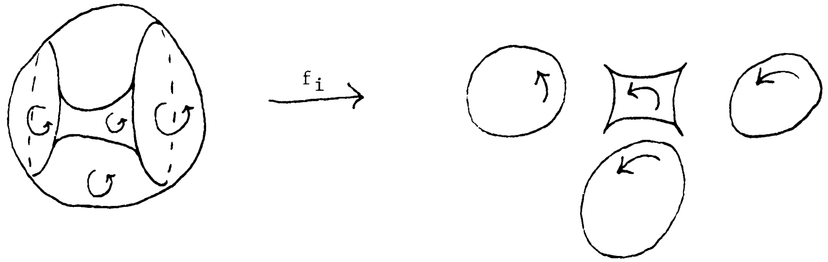

FIGURE 2

Clearly $A \circ d^2 \circ C^{-1}$ is the identity, and a computation with $X = S^2$ shows that the image of $W \circ d^2 \circ K^{-1}$ is $2\mathbf{Z}$. This concludes the proof of 2.5.

3.6 REMARK. The proof that $W\colon H_0(\Gamma^\infty; H_1M) \to \mathbf{Z}$ is an isomorphism follows §3, except for the substitution of the following lemma for Lemma 3.4.

3.7 LEMMA [**4**, THEOREM 3]. *$H_1|\Gamma^\infty| = 0$ and $H_2|\Gamma^\infty| = \mathbf{Z}$.*

4. Proof of 3.5. The real line $\mathbf{R}$, embedded in $\mathbf{R}^2$ as the x-axis is a submanifold of M. Let N be the discrete monoid of $\Gamma^\Omega\backslash M$-embeddings of the line; as a set

$$N = \left\{s\colon \mathbf{R} \to \left(\Gamma^\Omega\backslash M\right)_1 | D \circ s = \text{id and } R \circ s(\mathbf{R}) \subseteq \mathbf{R}\right\}.$$

The translates of $\mathbf{R}$ by $(\Gamma^\Omega\backslash M)_1$ generate the topology of M, so by Theorem 1.2(ii) of [**1**] there is a weak homotopy equivalence $BN \to |\Gamma^\Omega\backslash M|$. Let us show that $\pi_1 BN = \mathbf{Z}/2$.

Let K be the submonoid of N consisting of elements which preserve the orientation of the line; it is not hard to see that the exact sequence $K \to N \to \mathbf{Z}/2$ gives a homotopy fibration $BK \to BN \to B\mathbf{Z}/2$. Since $\pi_2 B\mathbf{Z}/2 = 0$, 3.5 will follow when we show that $\pi_1 BK = 0$.

So we show that the homomorphic image of K in any group is trivial. Now K is generated by elements k which are the identity section in some open set U (after [**7**], 3.1). But for any U there is an $m \in K$ such that $m(\mathbf{R}) \subseteq U$; therefore $km = m$ and k must map to the identity of any group. So all of K must map to the identity.

5. The spectral sequence 3.3. There is a spectral sequence for the action of a pseudogroup on a space, constructed in [**2**], which generalizes the spectral sequence for the action of a group on a space. The case at hand is an example of its application. We sketch the construction.

Let C be the discrete category whose objects are contactible open subsets of $\mathbf{R}^2$, with morphisms area and orientation preserving embeddings between open sets. Note that (as in [**8**, §1]) there is a weak homotopy equivalence between $|C|$ and $|\Gamma^\Omega|$.

Now recall the immersion $i\colon M \to \mathbf{R}^2$. Let $\underline{S}_q$ denote the complex of abelian group valued functors of C, where for U an open subset of $\mathbf{R}^2$, $\underline{S}_q U = S_q(i^{-1}U)$, where S_q is the usual singular q-chain functor. The spectral sequence for the complex $\underline{S}_q$ of functors satisfies 3.3. In particular, $E^2_{p0} = H_p|\Gamma^\Omega|$ because $i^{-1}U$ is connected if U is connected.

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