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SHARP ESTIMATES FOR LEBESGUE CONSTANTS

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ABSTRACT. Suppose $C \subset R^N$ is a closed convex bounded body containing 0 in its interior. If ∂C is sufficiently smooth with strictly positive Gauss curvature at each point, then, denoting by $L_{r,C}$ the Lebesgue constant relative to C, there exists a constant $A > 0$ such that $L_{r,C} \geqslant Ar^{(N-1)/2}$ for r sufficiently large. This complements the known result that there exists a constant B such that $L_{r,C} \leqslant Br^{(N-1)/2}$ for r sufficiently large.

1. Let R^N denote the N-dimensional euclidean space, T^N the N-dimensional torus (identified with the cube $Q_N = \{t \in R^N \colon -\frac{1}{2} \leqslant t_j < \frac{1}{2},\ j = 1,\ldots,N\}$), and Z^N the integer lattice of R^N. Throughout this paper we will assume $N \geqslant 2$. Suppose $C \subset R^N$ is a bounded closed convex body containing the origin in its interior and set $rC = \{x \in R^N \colon r^{-1}x \in C\}$ for every $r > 0$. To any such C we associate the Dirichlet kernel

$$D_{r,C}(t) = \sum \exp(2\pi imt), \qquad t \in T^N, \tag{1}$$

where the summation in (1) is extended to all $m \in rC \cap Z^N$. Sharp estimates for the Lebesgue constants

$$L_{r,C} = \int_{T^N} |D_{r,C}(t)|\, dt \tag{2}$$

are of interest in the study of norm convergence of multiple Fourier series. The aim of this paper is to prove there exists a positive constant A such that $L_{r,C} \geqslant Ar^{(N-1)/2}$ for r sufficiently large, whenever C satisfies the regularity conditions of C. Herz [**4**] (this complements the known result that there exists $B > 0$ such that $L_{r,C} \leqslant Br^{(N-1)/2}$ for r sufficiently large [**5**]). In fact our result is based on the asymptotic estimates of Herz for the Fourier transform of the characteristic function of C.

In the case where C is a ball, the above result was proved earlier by K. I. Babenko [**2**]. As the original paper is not readily accessible, and the proof given in [**1**] is complicated (as it deals with general eigenfunctions of the Laplacian), our theorem provides also a simple proof of the behaviour of the spherical Lebesgue constants.

As an application of our main result we extend to summations over C the result of [**3**] on the best conditions for norm convergence of multiple Fourier series.

2. In the following we will always suppose ∂C is of class C^k with $k = [(N-1)/2 + 4]$, and the Gauss curvature of ∂C is strictly positive at every point. We denote by

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H the supporting function of C and set for all $t \in R^N$:

$$g(t) = \rho^{-(N-1)/2} K(\theta)^{-1/2} \{\exp(2\pi i(H(t) - (N-1)/8))\}.$$

Here we made $t = \rho \cdot \theta$, where $\rho \geqslant 0$ and θ is a norm 1 vector. $K(\theta)$ is the Gauss curvature at the point $x \in \partial C$ corresponding to θ under the inverse of the normal mapping.

THEOREM 1. *If C is as above and $L_{r,C}$ are defined by* (2), *then there are positive constants A and B, depending only on C, such that as $r \to +\infty$,*

$$Ar^{(N-1)/2} \leqslant L_{r,C} \leqslant Br^{(N-1)/2}. \tag{3}$$

PROOF. The right-hand inequality in (3) is a particular case of a theorem of Yudin [**5**]. Hence we have to prove only the left-hand inequality. Arguing as in [**5** and **6**],

$$\int_{B_r} e^{2\pi i t s}\, ds = \left(\prod_{j=1}^{N} (\pi t_j)^{-1} \sin \pi t_j \right) D_{r,C}(t),$$

where B_r denotes the union of all cubes of edge 1 centered at the points $m \in rC \cap Z^N$. Hence

$$L_{r,C} \geqslant \int_{T^N} \left| \int_{B_r} e^{2\pi i t s}\, ds \right| dt. \tag{4}$$

Let c_r denote the characteristic function of rC, and b_r the characteristic function of B_r. Then $|b_r - c_r|$ is the characteristic function of the symmetric difference of B_r and rC. For a small positive ε (which will be chosen later) denote by F_ε the annulus $F_\varepsilon = \{x \in R^N\colon \varepsilon \leqslant |x| \leqslant 2\varepsilon\}$. Then, by (4),

$$L_{r,C} \geqslant \int_{F_\varepsilon} |\hat{c}_r(t)|\, dt - \int_{F_\varepsilon} |\hat{b}_r(t) - \hat{c}_r(t)|\, dt = I_1 - I_2. \tag{5}$$

The symmetric difference of B_r and rC is contained in the set of all points whose distance from ∂rC is smaller than $2N^{1/2}$. Hence, for big values of r (depending only on C) we have $\|b_r - c_r\|_2 \leqslant \text{const}\, r^{(N-1)/2}$. Therefore by the Schwarz inequality and the Plancherel theorem,

$$I_2 \leqslant M_1 \varepsilon^{N/2} \|b_r - c_r\|_2 \leqslant M_2 \varepsilon^{N/2} r^{(N-1)/2}, \tag{6}$$

where the constants M_1 and M_2 depend only on C.

On the other hand, Herz' asymptotic formula for $\hat{c}_r$ [**4**, Theorem 3] reads

$$2\pi i \hat{c}_r(t) = r^{(N-1)/2}\{(g(rt) - \overline{g(-rt)})|t|^{-1}\} + r^N O\left((r|t|)^{-(N+3)/2}\right) \tag{7}$$

where the error term depends only on C. Hence we have for, say, $r^{-1} \leqslant \varepsilon^2$,

$$\begin{aligned} r^N \int_{F_\varepsilon} O\left((r|t|)^{-(N+3)/2}\right) dt &\leqslant M_3 \varepsilon^{(N-3)/2} r^{(N-3)/2} \\ &\leqslant M_3 \varepsilon^{(N+1)/2} r^{(N-1)/2}, \end{aligned} \tag{8}$$

where the constant M_3 depends only on C. We set $f(\theta) = H(\theta) + H(-\theta)$. Then we get, by (7) and (8),

$$\begin{aligned} 2\pi I_1 &\geq r^{(N-1)/2} \int_{|\theta|=1} K(\theta)^{-1/2}\, d\theta \\ &\quad \cdot \int_{\varepsilon}^{2\varepsilon} \rho^{(N-3)/2} |1 - \exp - 2\pi i(r\rho f(\theta) - (N-1)/4)|\, d\rho \\ &\quad - M_3 \varepsilon^{(N+1)/2} r^{(N-1)/2} \\ &= \int_{|\theta|=1} K(\theta)^{-1/2} f(\theta)^{-1}\, d\theta \\ &\quad \cdot \int_{\varepsilon r f(\theta)}^{2\varepsilon r f(\theta)} x^{(N-3)/2} |1 - \exp - 2\pi i(x - (N+1)/4)|\, dx \\ &\quad - M_3 \varepsilon^{(N+1)/2} r^{(N-1)/2}. \end{aligned}$$

(We should remark that $f(\theta) > 0$ since 0 belongs to the interior of C.)

Since $|1 - \exp - 2\pi i(x - (N-1)/4)|$ is periodic, for large values of r ($r > 10\varepsilon^{-1}\max f(\theta)^{-1}$ will do) the inner integral is larger than

$$M_4 \sum_{j=h}^{k} j^{(N-3)/2} \geq M_5 \varepsilon^{(N-1)/2} f(\theta)^{(N-1)/2} r^{(N-1)/2},$$

where h and k are the integral parts of $r\varepsilon f(\theta) + 1$ and $2r\varepsilon f(\theta)$, respectively, and M_4, M_5 are constants independent of r and ε. It follows, for large values of r (depending only on C and ε), that

$$I_1 \geq M_6 \varepsilon^{(N-1)/2} r^{(N-1)/2} - M_3 \varepsilon^{(N+1)/2} r^{(N-1)/2}, \tag{9}$$

where M_6 is a constant depending only on C. Now, choose ε in such a way that

$$M_6 \varepsilon^{(N-1)/2} - M_3 \varepsilon^{(N+1)/2} - M_2 \varepsilon^{N/2} > 0. \tag{10}$$

Then the left-hand inequality in (3) follows from (6), (9) and (10).

The exact estimates of Theorem 1 allow an extension of Theorem A of **[3]**. Suppose B is either the space $C(T^N)$ or the space $L^1(T^N)$. Denote by $\omega_n(\delta, f)$ the nth modulus of smoothness of a function f in B. Then we have

THEOREM 2. *Suppose C is a closed, convex, bounded body satisfying the same assumptions as in Theorem* 1. *Then*

(a) *if* $f \in B$ and $\omega_n(\delta, f) = o(\delta^{(N-1)/2})$ *as* $\delta \to 0$, *for some* $n > (N-1)/2$, *then*

$$\|D_{r,C} * f - f\|_B \to 0 \quad \text{as } r \to \infty;$$

(b) *there exists a function $F \in B$ such that (for all $n > (N-1)/2$) $\omega_n(\delta, F) = O(\delta^{(N-1)/2})$ as $\delta \to 0$, but $D_{r,C} * F$ does not converge in norm as $r \to \infty$.*

In other words the condition stated in (a) is the best condition, in terms of moduli of smoothness, for the norm convergence over C of Fourier series in B. The proof of this theorem can be obtained by the same arguments as in **[3]**.

REFERENCES

1. S. A. Alimov and V. A. Il'in, *Conditions for the convergence of spectral decompositions that correspond to self-adjoint extensions of elliptic operators.* I, II, Differential Equations **7** (1971), 516–543; 651–667.

2. K. I. Babenko, *On the mean convergence of multiple Fourier series and the asymptotic behaviour of the Dirichlet kernel of the spherical means*, Preprint No. 52, Inst. Prikl. Mat. Akad. Nauk SSSR, Moscow, 1971.

3. D. I. Cartwright and P. M. Soardi, *Best conditions for the norm convergence of Fourier series*, J. Approx. Theory (to appear).

4. C. S. Herz, *Fourier transforms related to convex sets*, Ann. of Math. (2) **75** (1962), 81–92.

5. V. A. Yudin, *Behaviour of Lebesgue constants*, Mat. Zametki **17** (1975), 401–405.

6. A. A. Yudin and V. A. Yudin, *Discrete embedding theorems and Lebesgue constants*, Mat. Zametki **22** (1977), 381–394.

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