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# THE COMBINATORICS OF CERTAIN PRODUCTS

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ABSTRACT. A combinatorial interpretation for the coefficients in the expansion of $\Pi(1 + ux^jy^k)(1 - ux^jy^k)^{-1}$ is given.

**1. Introduction.** The coefficients $A(n; x, y)$, $B(n; x, y)$ and $C(n; x, y)$ defined by

$$\sum_{n\geqslant 0} \frac{A(n; x, y)u^n}{(x)_n(y)_n} = \frac{1}{(u; x, y)}, \tag{1.1}$$

$$\sum_{n\geqslant 0} \frac{B(n; x, y)u^n}{(x)_n(y)_n} = (-u; x, y), \tag{1.2}$$

$$\sum_{n\geqslant 0} \frac{C(n; x, y)u^n}{(x)_n(y)_n} = \frac{(-u; x, y)}{(u; x, y)}, \tag{1.3}$$

where

$$(x)_n = (1 - x)(1 - x^2) \cdots (1 - x^n), \qquad (x)_0 = 1, \tag{1.4}$$

$$(u; x, y) = \prod_{j,\, k\geqslant 0} (1 - ux^jy^k), \tag{1.5}$$

have been considered by a number of mathematicians. Carlitz **[2]** was the first to demonstrate that these coefficients are polynomials in $x$ and $y$ with positive integral coefficients and provided closed formulas for calculating them. In a subsequent paper Roselle **[6]** gave combinatorial interpretations for $A(n; x, y)$ and $B(n; x, y)$. More recently, generalizations of (1.1) and (1.2) have appeared in **[3, 4, 5]** and **[4]**, respectively.

A natural question, which was suggested to me by Dominique Foata, arises from Roselle's work: Is there a combinatorial interpretation for $C(n; x, y)$? This note provides one in terms of the already known interpretations of $A(n; x, y)$ and $B(n; x, y)$ and in terms of certain statistics defined on bicolored permutations.

**2. Roselle's work.** The polynomials $A(n; x, y)$ and $B(n; x, y)$ may be interpreted as the generating functions for various statistics defined on the group $G(n)$ consisting of permutations of $\{1, 2, \ldots, n\}$. To be precise, if the major index of a permutation $\sigma \in G(n)$, denoted $m(\sigma)$, is defined to be the sum of the elements in the set

$$\{j\colon \sigma(j) > \sigma(j + 1),\ 1 \leqslant j \leqslant n - 1\} \tag{2.1}$$

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and the $i$-major and co-$i$-major indices are defined, respectively, to be

$$i(\sigma) = m(\sigma^{-1}), \tag{2.2}$$

$$c(\sigma) = \binom{n}{2} - i(\sigma), \tag{2.3}$$

then, as Roselle [6] and Foata [4] have demonstrated,

$$A(n; x, y) = \sum_{\sigma \in G(n)} x^{m(\sigma)} y^{i(\sigma)}, \tag{2.4}$$

$$B(n; x, y) = \sum_{\sigma \in G(n)} x^{m(\sigma)} y^{c(\sigma)}. \tag{2.5}$$

**3. Bicolored permutations.** Since Carlitz [2] showed that $C(n; 1, 1) = 2^n \cdot n!$, it is clear that $G(n)$ will not provide a combinatorial structure for interpreting $C(n; x, y)$. For this reason, one is led to consider the set of bicolored permutations $BG(n)$, consisting of words $b = b(1)b(2) \cdots b(n)$ obtained by coloring each letter of some permutation $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n) \in G(n)$ either red (indicated by underlining the letter) or blue (letter not underlined). For instance,

$$b = 3\,\underline{2}\,7\,\underline{4}\,\underline{1}\,\underline{6}\,5 \in BG(7). \tag{3.1}$$

Note that there are $2^n$ different colorings of each $\sigma \in G(n)$.

Now, possessing a combinatorial set with the correct cardinality, all that remains is to define some appropriate statistics on $BG(n)$ for which $C(n; x, y)$ is the generating function. To this end, let $W(n)$ be the set of words $w = w(1)w(2) \cdots w(n)$ of length $n$ with letters $w(i) \in \{0, 1\}$, and let $|w|$ denote the number of letters equal to 1 in $w$. The reduction of a permutation $\gamma$ of $\{a_1 < a_2 < \cdots < a_n\}$ is obtained by replacing $a_i$ in $\gamma$ by $i$ for $1 \leqslant i \leqslant n$. For instance, the reduction of the permutation $\gamma = 2\,4\,1\,6$ of $\{1 < 2 < 4 < 6\}$ is $2\,3\,1\,4$.

Each $b \in BG(n)$ is now assigned to a 4-tuple $(w, v, \theta, \alpha)$ where $w, v \in W(n)$ with $|w| = |v|$, $\theta \in G(|w|)$ and $\alpha \in G(n - |w|)$ according to the following rules:

$$w(i) = \begin{cases} 1 & \text{if } b(i) \text{ is red,} \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

$$v(i) = \begin{cases} 1 & \text{if } i \text{ is red in } b, \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

$$\theta = \text{reduction of the red subpermutation of } b, \tag{3.4}$$

$$\alpha = \text{reduction of the blue subpermutation of } b. \tag{3.5}$$

Roughly speaking, $w$ indicates which positions of $b$ are red, $v$ which elements of $\{1, 2, \ldots, n\}$ are placed in the red positions, $\theta$ how the red letters are arranged in the red positions, and $\alpha$ how the blue letters are placed in the remaining positions. For instance, the bicolored permutation $b$ of (3.1) corresponds to the 4-tuple $(w, v, \theta, \alpha)$ where $w = 0\,1\,0\,1\,1\,1\,0$, $v = 1\,1\,0\,1\,0\,1\,0$, $\theta = 2\,3\,1\,4$ and $\alpha = 1\,3\,2$. Note that the map $b \to (w, v, \theta, \alpha)$ is a bijection between $BG(n)$ and the set of such 4-tuples.

Finally, for the bicolored permutation $b \to (w, v, \theta, \alpha)$ the appropriate statistics are defined by

$$M(b) = m(\theta) + m(\alpha) + m(w), \tag{3.6}$$

$$I(b) = i(\theta) + c(\alpha) + m(v), \tag{3.7}$$

where $m(w)$ and $m(v)$ are defined in exactly the same way as the major index of a permutation.

**4. The interpretation of** $C(n;\, x,\, y)$. It is now possible to show that

$$C(n;\, x,\, y) = \sum_{b \in BG(n)} x^{M(b)} y^{I(b)}. \tag{4.1}$$

First, as Carlitz [**2**] demonstrated, identities (1.1)–(1.3) imply

$$C(n;\, x,\, y) = \sum_{k \geqslant 0} \binom{n}{k}_x \binom{n}{k}_y A(k;\, x,\, y) B(n-k;\, x,\, y), \tag{4.2}$$

where the $x$-binomial coefficient is defined by

$$\binom{n}{k}_x = \frac{(x)_n}{(x)_k (x)_{n-k}}. \tag{4.3}$$

Second, the fact that (see [**1**, p.40])

$$\sum x^{m(w)} = \binom{n}{k}_x, \tag{4.4}$$

where the summation is over $w \in W(n)$ with $|w| = k$, along with (2.4), (2.5), (3.6) and (3.7) allow the calculation

$$\begin{aligned} \sum_{b \in BG(n)} x^{M(b)} y^{I(b)} &= \sum_{(w,\, v,\, \theta,\, \alpha)} x^{m(\theta) + m(\alpha) + m(w)} y^{i(\theta) + c(\alpha) + m(v)} \\ &= \sum_{k \geqslant 0} \sum_{|w| = k} x^{m(w)} \sum_{|v| = k} y^{m(v)} \sum_{\theta \in G(k)} x^{m(\theta)} y^{i(\theta)} \sum_{\alpha \in G(n-k)} x^{m(\alpha)} y^{c(\alpha)} \\ &= \sum_{k \geqslant 0} \binom{n}{k}_x \binom{n}{k}_y A(k;\, x,\, y) B(n-k;\, x,\, y). \end{aligned} \tag{4.5}$$

Identities (4.2) and (4.5) imply (4.1).

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