PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 4, December 1983

CYCLIC ALGEBRAS OF SMALL EXPONENT

J.-P. TIGNOL[1]

ABSTRACT. We prove that every cyclic algebra of exponent n and degree mn over a field containing a primitive nth root of unity is similar to a tensor product of at most m symbols of degree n.

1. Introduction. Let F be a field containing a primitive nth root of unity ω. A central simple F-algebra of degree n (i.e. of dimension n^2) is called a *symbol* if it is generated by two elements x, y subject to the relations $x^n \in F^*$, $y^n \in F^*$ and $yx = \omega xy$ (compare [**3**, §15]). Merkurjev and Suslin [**2**] have recently proved that every finite-dimensional central simple F-algebra of exponent n (i.e. whose similarity class has order n in the Brauer group $\mathrm{Br}(F)$) is similar to a tensor product of symbols of degree n.

The aim of this note is to give a simple proof of this theorem for *cyclic* algebras, i.e. for central simple algebras which contain a cyclic extension of the center as a maximal commutative subalgebra.

THEOREM. *Let F be a field containing a primitive nth root of unity. Every cyclic F-algebra of exponent n and degree mn is similar to a tensor product of at most m symbols of degree n.*

If K is an extension of a field F, we denote by $\mathrm{Br}(K/F)$ the kernel of the natural map from $\mathrm{Br}(F)$ to $\mathrm{Br}(K)$ and by $\mathrm{Br}_n(K/F)$ the subgroup of $\mathrm{Br}(K/F)$ which is killed by n.

2. LEMMA. *Let K/F be a cyclic field extension and let L be an intermediate field. Let $n = [K: L]$. Then, the image of the corestriction map*

$$\mathrm{Cor}_{L/F}\colon \mathrm{Br}(K/L) \to \mathrm{Br}(K/F)$$

is $\mathrm{Br}_n(K/F)$.

PROOF. Let G be the Galois group of K over F and let χ be a generator of the group $\mathrm{Hom}(G, \mathbf{Q}/\mathbf{Z})$ of characters of G. By [**6**, Corollary 2, p. 211], every element of $\mathrm{Br}(K/F)$ is of the form (χ, a) for some $a \in F^*$. (If we denote by σ the generator of G such that $\chi(\sigma) = [K: F]^{-1} \pmod{\mathbf{Z}}$, then (χ, a) is the similarity class of the cyclic algebra (K, σ, a), with the notations of [**1**, p. 74].) If (χ, a) is killed by n, then $(n\chi, a) = 0$.

Received by the editors December 29, 1982.
1980 *Mathematics Subject Classification*. Primary 16A16; Secondary 13A20, 12G05.
Key words and phrases. Cyclic algebras, Brauer group of fields.

[1] The author wishes to thank M. Ojanguren for calling [**4**] to his attention.

Since the kernel of $n\chi$ is the Galois group of K over L, we then have $a = N_{L/F}(t)$ for some $t \in L^*$, by [**6**, Corollary 1, p. 211], whence $(\chi, a) = \mathrm{Cor}_{L/F}(\mathrm{Res}_{L/F}\chi, t)$, by the "projection formula" (see [**6**, p. 212 or **7**, Proposition 4.3.7]). This proves that the image of $\mathrm{Br}(K/L)$ by the corestriction map contains $\mathrm{Br}_n(K/F)$. The converse is clear, since the exponent of $\mathrm{Br}(K/L)$ divides $[K : L] = n$. Q.E.D.

3. Proof of the Theorem. Let K be a cyclic extension of F, of rank mn. The Lemma shows that every element in $\mathrm{Br}_n(K/F)$ is the corestriction of some element in $\mathrm{Br}(K/L)$, where L is the (unique) extension of F of codimension n in K. Since L contains a primitive root of unity, every element in $\mathrm{Br}(K/L)$ is the similarity class of a symbol of degree n and, since $[L : F] = m$, the corestriction of any symbol of degree n over L is similar to the tensor product of at most m symbols of degree n over F, by a theorem of Rosset and Tate [**4**, §3, Corollary 1]. Q.E.D.

4. Remarks. (1) If n is a product of relatively prime integers $n = n_1 \cdots n_r$ then, by [**1**, Theorem 7.20], every cyclic algebra A of exponent n is isomorphic to a tensor product $A \simeq A_1 \otimes \cdots \otimes A_r$, where A_i is a cyclic algebra of exponent n_i for $i = 1, \dots, r$. The Theorem above can thus be applied separately to $A_1, \dots, A_r$; this yields a better bound for the number of factors in a decomposition of A as a tensor product of symbols (up to similarity). If n is a power of a prime integer, it is not known whether the bound is the best possible. (It is obviously so for $m = 1$ or 2.)

(2) For $n = 2$, the Theorem above has also been proved by Rowen [**5**, Theorem 3.7], under the extra hypothesis that F contains a primitive $2m$th root of unity. His techniques are different and do not yield a bound on the number of symbols.

REFERENCES

1. A. A. Albert, *Structure of algebras*, Amer. Math. Soc. Colloq. Publ., vol. 24, Amer. Math. Soc., Providence, R. I., 1939.

2. A. S. Merkurjev and A. A. Suslin, *K-cohomology of Severi-Brauer varieties and norm residue homomorphism*, preprint, 1981.

3. J. Milnor, *Introduction to algebraic K-theory*, Ann. of Math. Studies, no. 72, Princeton Univ. Press, Princeton, N. J., 1971.

4. S. Rosset and J. Tate, *A reciprocity law for K_2-traces*, preprint, 1982.

5. L. H. Rowen, *Central simple algebras with involution viewed through centralizers*, J. Algebra **63** (1980), 41–55.

6. J.-P. Serre, *Corps locaux*, Hermann, Paris, 1968.

7. E. Weiss, *Cohomology of groups*, Academic Press, New York, 1969.

INSTITUT DE MATHÉMATIQUE, UNIVERSITÉ CATHOLIQUE DE LOUVAIN, B-1348 LOUVAIN-LA-NEUVE, BELGIUM