

EXTREME POINTS IN FUNCTION SPACES

DIRK WERNER

ABSTRACT. We show that the extreme points of the unit ball of $C(K, L^1(\mu))$ (K compact Hausdorff, (Ω, Σ, μ) arbitrary) are precisely the functions with extremal values. The result is applied to characterize the extreme points of the unit ball of certain spaces of compact operators.

1. Introduction and notation. The object of this paper is to characterize the extreme points of the unit ball of certain spaces of vector-valued continuous functions. A natural conjecture about such extremal functions is that all the values they assume are extremal points of the unit ball of the range space. This conjecture is obviously true for strictly convex and $C(K)$ -type range spaces, but it is false in general (for counterexamples see [1 or 3]). Our main theorem will establish the above mentioned characterization for spaces of $L^1(\mu)$ -valued functions. We shall apply this result to operator spaces, thereby deducing a theorem due to Morris and Phelps on extremal operators.

Our notation is standard. K and L denote compact Hausdorff spaces, X and Y Banach spaces. The extreme points of the unit ball of X are called extremal in X . $C(K, X)$ stands for the Banach space of X -valued continuous functions on K , equipped with the norm $\|f\| = \sup_{k \in K} \|f_k\|$, the value of $f \in C(K, X)$ at $k \in K$ being denoted by f_k instead of $f(k)$. $\mathcal{K}(X, Y)$ is the space of compact operators from X into Y with the usual operator norm. Finally, we write $\mathbf{1}_A$ for the characteristic function of a set A and r for the constant function with value r . The field of scalars may be real or complex.

2. Extreme points in $C(K, L^1(\mu))$. We are going to prove the following theorem.

2.1. THEOREM. *Suppose (Ω, Σ, μ) is an arbitrary measure space, and f is extremal in $C(K, L^1(\mu))$. Then f_k is extremal in $L^1(\mu)$ for all $k \in K$.*

PROOF. First note that $\|f_k\| = 1$ for all $k \in K$. Assume f_{k_0} fails to be extremal for some $k_0 \in K$. Applying a selection theorem of Michael's (cf. [4]), we shall represent f as a nontrivial convex combination of norm-one functions. This contradiction furnishes the proof of 2.1.

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Of course, we need a custom-made description of the extreme points of the unit ball of $L^1(\mu)$. Recall $u \in L^1(\mu)$ is extremal if and only if $u = r \mathbf{1}_A / \mu(A)$ (r a scalar with modulus 1, $A \in \Sigma$ an atom with $\mu(A) < \infty$). Equivalently, $u \in L^1(\mu)$ with $\|u\| = 1$ is not extremal if and only if there exists $h \in L^\infty(\mu)$, $0 \leq h \leq 2$, $\int h \cdot |u| d\mu = 1$ with $h \cdot u \neq u$. (This description is easy to prove.)

Thus, one is led to consider the set-valued function $S: K \rightarrow \mathcal{P}(L^\infty(\mu))$, $S(k) := \{h/h \in L^\infty(\mu), 0 \leq h \leq 2, \int h \cdot |u| d\mu = 1\}$. Now our assumption simply means: there exists $h_0 \in S(k_0)$ such that $h_0 \cdot f_{k_0} \neq f_{k_0}$. We are going to prove that S admits a norm-continuous selection s (i.e. a continuous function $s: K \rightarrow L^\infty(\mu)$ with $s_k \in S(k)$ for all $k \in K$), which assumes the prescribed value $s_{k_0} = h_0$. This is accomplished in Lemma 2.2.

Continuing the proof of 2.1, with the aid of 2.2, we define $g_k := s_k \cdot f_k$, $g'_k := (2 - s_k) \cdot f_k$ ($k \in K$). Hölder's inequality yields $g_k, g'_k \in L^1(\mu)$, and $s_k \in S(k)$ gives $\|g_k\| = \|g'_k\| = 1$, for all $k \in K$. Finally, the continuity of f, s and of multiplication $L^\infty(\mu) \times L^1(\mu) \rightarrow L^1(\mu)$ shows that g and g' are continuous functions. Therefore $f = \frac{1}{2}(g + g')$ is a convex combination of norm-one functions in $C(K, L^1(\mu))$. But this is a nontrivial convex combination since $f_{k_0} \neq h_0 \cdot f_{k_0} = g_{k_0}$, and f is not extremal.

We still have to prove

2.2. LEMMA. S admits a continuous selection s with $s(k_0) = h_0$.

PROOF. In view of [4, Examples 1.3 and 1.3*, Theorem 3.2''] we have to show

- (i) $S(k)$ is nonvoid, convex and norm-closed for all $k \in K$,
- (ii) S is lower semicontinuous, meaning for open $U \subset L^\infty(\mu)$, $\{k/k \in K, S(k) \cap U \neq \emptyset\}$ is open.

(i) is obvious, and for the proof of (ii) we show $F_U := \{k/k \in K, S(k) \cap U \neq \emptyset\}$ is closed for open $U \subset L^\infty(\mu)$. Let (k_i) be a net in F_U , such that $k_i \rightarrow k^* \in K$.

Claim. $k^* \in F_U$. Otherwise, there would be $h^* \in S(k^*) \cap U$.

Choose $\varepsilon > 0$ (w.l.o.g. $\varepsilon < \frac{1}{2}$) with $\{H/H \in L^\infty(\mu), \|H - h^*\| < \varepsilon\} \subset U$. $k_i \rightarrow k^*$ and the continuity of f yield $f_{k_i} \rightarrow f_{k^*}$, therefore $|f_{k_i}| \rightarrow |f_{k^*}|$ and $d_i := \int h^* \cdot |f_{k_i}| d\mu \rightarrow \int h^* \cdot |f_{k^*}| d\mu = 1$. In particular, there exists i_0 such that $|d_i - 1| < \varepsilon/3$ for all $i > i_0$. For the following discussion, fix $i > i_0$, and construct an element of $S(k_i) \cap U$ in contrast to $k_i \in F_U$. We must consider the cases (a) $d_i \geq 1$, and (b) $d_i < 1$, separately (note $d_i \in \mathbf{R}$).

Case (a). It is easy to see that $h_i := d_i^{-1} \cdot h^* \in S(k_i)$. Our choice of ε yields $h_i \in U$ because of $\|h_i - h^*\| = |d_i^{-1} - 1| \cdot \|h^*\| \leq (d_i - 1)/d_i \cdot 2 < \varepsilon$.

Case (b). Consider $E := \{w/w \in \Omega, h^*(w) \leq 3/2\}$, a measurable subset of Ω . Of course, E depends on the realization of the equivalence class h^* , but this is not substantial for our purpose. Define $A: [0, \frac{3}{2}] \rightarrow \mathbf{R}$ by $A(r) := \int (h^* + r \mathbf{1}_E) \cdot |f_{k_i}| d\mu$. A is a continuous affine function with $A(0) = d_i < 1$, and $A(\frac{3}{2}) \geq \frac{3}{2} \int |f_{k_i}| d\mu = \frac{3}{2}$. Therefore, there exists $r_0 \in [0, \frac{3}{2}]$ with $A(r_0) = 1$. Calculate r_0 :

$$\frac{r_0}{3/2} = \frac{A(r_0) - A(0)}{A(3/2) - A(0)} = \frac{1 - d_i}{A(3/2) - d_i} < \frac{\varepsilon/3}{3/2 - 1} = \frac{2}{3}\varepsilon,$$

consequently $r_0 < \varepsilon (< \frac{1}{2})$.

We now put $h_i := h^* + r_0 \mathbf{1}_E$. $A(r_0) = 1$ and the above inequality guarantee $h_i \in S(k_i) \cap U$. This completes the demonstration of our claim and thus of Lemma 2.2.

An immediate corollary of Theorem 2.1 is

2.3. COROLLARY. *Suppose (Ω, Σ, μ) is a purely nonatomic measure space. Then the unit ball of $C(K, L^1(\mu))$ does not possess any extreme point.*

Next we apply Theorem 2.1 to Banach spaces of compact operators. It is well known that the space of continuous (resp. compact) linear operators from X into $C(K)$ is isometrically isomorphic to the space of weak*- (resp. norm-)continuous functions from K into X^* via the isometry $T \mapsto (k \mapsto T^*(\delta_k))$ (δ_k : Dirac measure at k), cf. [2, p. 490]. Morris and Phelps [5] call a continuous linear operator $T: X \rightarrow Y$ "nice", if the adjoint T^* maps extreme functionals onto extreme functionals. If, in the case $Y = C(K)$, we regard T as an X^* -valued function τ on K , T is nice if and only if τ assumes only extremal values. Of course, every nice operator is extremal. Employing measure theoretic arguments, several authors investigate the problem: Is every extreme operator from $C(L)$ into $C(K)$ nice? (For examples, see [1, 5 or 6].) Theorem 2.1 enables us to generalize a result of Morris and Phelps.

2.4. THEOREM. *If X is a Lindenstrauss space, then every extreme point of the unit ball of $\mathcal{K}(X, C(K))$ is a nice operator.*

PROOF. Regarding an extreme operator $T \in \mathcal{K}(X, C(K))$ as an extremal norm-continuous function $\tau: K \rightarrow X^* \cong L^1(\mu)$, Theorem 2.1 shows that τ assumes only extremal values, which means T is nice.

As $C(L)$ is a Lindenstrauss space, we have proved

2.5. COROLLARY (MORRIS AND PHELPS). *Every extreme point of the unit ball of $\mathcal{K}(C(L), C(K))$ is a nice operator.*

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