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ON A PROBLEM OF BANACH

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ABSTRACT. Assuming the continuum hypothesis, we obtain a translation invariant version of the following result of E. Grzegorek: There are two countably generated σ-algebras on the interval [0, 1] such that both carry a nonatomic countably additive probability measure, but the σ-algebra generated by their union does not carry any such measure.

Banach [**1**] asked if there are two countably generated σ-algebras on the interval $[0, 1)$ such that both carry a nonatomic countably additive probability measure, but the σ-algebra generated by their union does not carry any such measure, i.e. is "nonmeasurable". This problem was answered positively by E. Grzegorek in [**2**] assuming Martin's Axiom and in [**3**] without any additional set-theoretic assumptions. The result in [**2**] satisfies the additional condition that both σ-algebras extend the σ-algebra of Borel subsets of [0, 1]. This additional condition does not hold for the algebras obtained in [**3**]. Indeed, it is unlikely that such an improvement is possible without some additional set theoretic assumption, since this would imply that there is no real-valued measurable cardinal below the continuum. In the present paper we obtain an analog of the result of [**2**] in a translation invariant setting—assuming the continuum hypothesis. We do not know whether Martin's axiom suffices.

We shall work on $[0, 1)$, mod 1. The σ-algebras in the conclusion of the Theorem have as their underlying set the entire $[0, 1)$. However, the proof of the Theorem also uses σ-algebras whose underlying set is a proper subset of $[0, 1)$. The concepts we introduce should be viewed within the context of this more general class of σ-algebras. We say that a σ-algebra is measurable if it carries a σ-additive nonatomic probability measure. A σ-algebra $\mathcal{C}$ is translation invariant if for all $A \in \mathcal{C}$ and $a \in [0, 1]$, $A + a \in \mathcal{C}$. Clearly, $\mathcal{C}$ is translation invariant if it is closed under translates of members of a generating subfamily. Hence, the union of two translation invariant σ-algebras generates a translation invariant σ-algebra.

Obviously, the underlying set of a translation invariant σ-algebra has to be the entire $[0, 1)$. It will be convenient to work with a less restrictive concept of almost translation invariance. A σ-algebra $\mathcal{C}$ is almost translation invariant if for all $A \in \mathcal{C}$ and all $x \in [0, 1)$, there is some $B \in \mathcal{C}$ such that $|(A + x) \triangle B| \leq \aleph_0$. We shall also call a set A almost translation invariant if for all $x \in [0, 1)$, $|(A + x) \triangle A| \leq \aleph_0$.

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The σ-algebra of the relative Borel subsets of an almost translation invariant set is easily seen to be almost translation invariant. An almost translation invariant σ-algebra containing all points (and thus all enumerable sets) is translation invariant.

We shall denote the Lebesgue measure by λ and the Lebesgue outer measure by λ^*.

Our main result is the following.

THEOREM. *Assume the continuum hypothesis. Then there exist countably generated σ-algebras $\mathcal{Q}_1, \mathcal{Q}_2$ of subsets of the interval* $[0, 1)$ *and probability measures μ_1, μ_2 on $\mathcal{Q}_1, \mathcal{Q}_2$, respectively, such that*:

(i) $\mathcal{Q}_1, \mathcal{Q}_2$ *both contain all Borel sets and are translation invariant*;

(ii) μ_1, μ_2 *both extend the Lebesgue measure and are translation invariant*;

(iii) *there is no nonatomic probability measure on any σ-algebra containing $\mathcal{Q}_1 \cup \mathcal{Q}_2$.*

LEMMA 1 [**4, 6**]. *Assume* CH. *Then there is an $X \subset [0, 1)$ such that $\lambda^*(X) = \lambda^*([0, 1) \setminus X) = 1$ and both X and $[0, 1) \setminus X$ are almost translation invariant.*

PROOF. Let $\{x_\xi \colon \xi < \omega_1\}$ be a Hamel basis for $[0, 1)$. Let Q_ξ be the subspace spanned by $\{x_\eta \colon \eta < \xi\}$ (over the rationals) and let $P_\xi = Q_{\xi+1} \setminus Q_\xi$. Clearly $|Q_\xi| \leqslant \aleph_0$. Thus P_ξ consists of those elements of $Q_{\xi+1}$ whose representation uses x_ξ in a nontrivial way. Hence if $x \in Q_\xi$, then $P_\xi + x = P_\xi$. It now follows easily that for all $A \subseteq \omega_1$, $\bigcup \{P_\xi \colon \xi \in A\}$ is almost translation invariant.

Let $\{F_\alpha \colon \alpha < \omega_1\}$ be an enumeration of all closed subsets of $[0, 1)$ with $\lambda(F_\alpha) > 0$. We can build inductively $A \subset \omega_1$ such that for all $\alpha \in \omega_1$, there are $\xi \in A$, $\eta \in \omega_1 \setminus A$ such that both $P_\xi \cap F_\alpha$ and $P_\eta \cap F_\alpha$ are not empty. Now $X = \bigcup \{P_\xi \colon \xi \in A\}$ as is required.

LEMMA 2 [**5**]. *Assume* CH. *Let $\mathcal{F} \subset \mathcal{P}([0, 1))$, $|\mathcal{F}| = \aleph_1$. Then there exists a countably generated σ-algebra $\mathcal{Q}$ on* $[0, 1)$ *such that $\mathcal{F} \subset \mathcal{Q}$.*

PROOF. We can suppose $\mathcal{F} \subset \mathcal{P}(\omega_1)$, $\mathcal{F} = \{F_\alpha \colon \alpha < \omega_1\}$. Set

$$F = \bigcup \{\{\alpha\} \times F_\alpha \colon \alpha < \omega_1\}.$$

Now by [**5, 7**], F belongs to the σ-algebra generated by some sets $A_n \times B_n$, $n \in \omega$, where $A_n, B_n \subset \omega_1$. It is easy to see that $\mathcal{F}$ is included in the σ-algebra generated by the B_n's.

LEMMA 3. *Assume* CH. *Then for every uncountable $X \subset [0, 1)$, there is a nonmeasurable countably generated σ-algebra $\mathcal{Q}$ on X containing all Borel subsets of X.*

PROOF. By [**7**], there is an $L \subset [0, 1)$ such that $|L| = \aleph_1$ and $|L \cap N| \leqslant \aleph_0$ for every meager N. It follows easily from the fact that every Borel measure on $[0, 1)$ concentrates on a meager set that the σ-algebra of Borel subsets of L is nonmeasurable.

Since the properties involved are preserved under bijections, the σ-algebra constructed on L can be transplanted to any set $X \subset [0, 1)$ of cardinality $\aleph_1$. The resulting σ-algebra contains singletons and augmented by adding the Borel subsets of X remains nonmeasurable.

Proof of the Theorem. Let X be as in Lemma 1 and $X' = [0, 1)\setminus X$. Let $\mathfrak{N}_0$ be a nonmeasurable countably generated σ-algebra on X containing the Borel subsets of X (see Lemma 3). Let $T(\mathfrak{N}_0) = \{(x + S) \cap X: x \in [0, 1), S \in \mathfrak{N}_0\}$. Then $|T(\mathfrak{N}_0)| = \aleph_1$, and thus by Lemma 2, there is a countably generated σ-algebra $\mathfrak{N}_1 \supset T(\mathfrak{N}_0)$. Defining $T(\mathfrak{N}_1)$ analogously to $T(\mathfrak{N}_0)$, we can find countably generated $\mathfrak{N}_2 \supset T(\mathfrak{N}_1)$, and so on. Let $\mathfrak{N}$ be the σ-algebra generated by $\bigcup\{\mathfrak{N}_n: n \in \omega\}$. Then $\mathfrak{N}$ is countably generated and nonmeasurable. It also follows easily from the almost translation invariance of X that $\mathfrak{N}$ is almost translation invariant. Let $\mathfrak{N}'$ be obtained analogously on X' and let $\mathfrak{M}$ and $\mathfrak{M}'$ be the σ-algebras of Borel subsets of X and X', respectively. Thus $\mathfrak{M}$ and $\mathfrak{M}'$ are almost translation invariant and so are $\mathfrak{A}_1 = \{A \cup A': A \in \mathfrak{M}, A' \in \mathfrak{N}'\}$ and $\mathfrak{A}_2 = \{A \cup A': A \in \mathfrak{N}, A' \in \mathfrak{M}'\}$. But $\mathfrak{A}_1$ and $\mathfrak{A}_2$ contain all points, hence, they are translation invariant.

We now define μ_i on $\mathfrak{A}_i$ ($i = 1, 2$), as follows. Let $A \cup A' \in \mathfrak{A}_1$ where: $A \in \mathfrak{M}$, $A' \in \mathfrak{N}'$. Set $\mu_1(A \cup A') = \lambda^*(A)$. For $A \cup A' \in \mathfrak{A}_2$ where $A \in \mathfrak{N}$, $A' \in \mathfrak{M}'$, we set $\mu_2(A \cup A') = \lambda^*(A')$. (i) and (ii) now easily follow. (We remark that λ^*, restricted to the Borel subsets of an arbitrary set of outer measure one, is a countably additive probability measure.)

Finally, let $\mathfrak{A}$ be the least σ-algebra containing $\mathfrak{A}_1 \cup \mathfrak{A}_2$. If μ is a nonatomic probability measure on $\mathfrak{A}$, then either $\mu(X) > 0$ or $\mu(X') > 0$. Without loss of generality let $\mu(X) > 0$. Since $\mathfrak{N} \subset \mathfrak{A}$ and $X \in \mathfrak{N}$, μ restricted to $\mathfrak{N}$ is a nonatomic finite nontrivial measure on $\mathfrak{N}$. This is a contradiction since $\mathfrak{N}$ is nonmeasurable. Hence, (iii) holds and the Theorem is proved.

References

1. S. Banach, *Sur les suites d'ensembles excluant l'existence d'une mesure* (Note posthume avec préface et commentaire de E. Marczewski), Colloq. Math. **1** (1948), 103–108.

2. E. Grzegorek, *Remarks on σ-fields without continuous measures*, Colloq. Math. **39** (1978), 73–75.

3. ______, *Solution of a problem of Banach on σ-fields without continuous measure*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **28** (1980), 7–10.

4. S. Kakutani and J. Oxtoby, *Construction of a non-separable invariant extension of the Lebesgue measure space*, Ann. of Math. (2) **52** (1950), 580–590.

5. B. V. Rao, *On discrete Borel spaces and projective sets*, Bull. Amer. Math. Soc. **75** (1969), 614–617.

6. W. Sierpiński, *Sur les translations des ensembles linéaires*, Fund. Math. **19** (1932), 22–28.

7. ______, *Hypothese du continu*, Chelsea, New York, 1956.

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