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A NOTE ON THE ROTATION NUMBER OF POINCARÉ

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ABSTRACT. The present note gives a formula relating the rotation number, the conjugating function and the vector field for a flow on a torus. Furthermore, in a particular case, it gives a formula such that the rotation number ρ can be computed only by means of the vector field $f(x, y)$.

1. Introduction. Consider the differential equation defined on a torus T^2

$$dy/dx = f(x, y), \tag{1}$$

and suppose that it satisfies the following conditions:

H1. $f(x, y)$ is continuous for all x, y in the xy-plane and it has continuous first partial derivatives with respect to x, y.

H2. $f(x + 1, y) = f(x, y + 1) = f(x, y)$ for all x, y.

The condition H1 implies that there is a unique solution of (1) through any given point (x_0, η) in the xy-plane. Let $y = \varphi_1(x, x_0, \eta)$ be the solution of (1) which satisfies $\varphi_1(x_0, x_0, \eta) = \eta$. Since every solution of (1) exists on $-\infty < x < +\infty$, it follows that every integral curve on T^2 must cross the meridian C given by $x = 0$ and therefore it is sufficient to take the initial values as $(0, \eta)$. Let $y = \varphi(x, \eta) \overset{\text{def}}{=} \varphi_1(x, 0, \eta)$ designate the solution of (1) satisfying the initial condition $y(0) = \varphi(0, \eta) = \varphi_1(0, 0, \eta) = \eta$. If $\psi(\eta) = \varphi(1, \eta)$, then $\eta \to \psi(\eta)$ defines a topological mapping of the real line onto itself. In some cases (for example, in the ergodic case), it is known that there exists a monotone increasing function $h(\xi)$ which is continuous in ξ and satisfies

$$h(\xi + 1) = h(\xi) + 1, \tag{2}$$

$$h(\psi(\xi)) = h(\xi) + \rho, \tag{3}$$

for all $\xi \in (-\infty, +\infty)$, where ρ is the rotation number. Such a function $h(\xi)$ is called a conjugating function of (1).

It is easy to see that when η varies on $(-\infty, +\infty)$, one may obtain the general solution of (1) which is denoted by

$$y = \varphi_1(x, 0, \eta). \tag{4}$$

From the uniqueness of solutions and (4), we obtain

$$\eta = \varphi_1(0, x, y) \overset{\text{def}}{=} \eta(x, y). \tag{5}$$

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In other words, the function $\eta(x, y)$ is a general integral of (1) in the xy-plane. By H1, $\eta(x, y)$ has continuous first partial derivatives with respect to x, y.

As is well known, it is important to study the rotation number ρ for the study of the global structure of integral curves of the equation (1). However, the computation of the rotation number is still an open question. In this paper, we obtain an identity relating the rotation number, the conjugating function and the vector field for the equation (1). In general, the formula does not yield a formula for the rotation number ρ in terms of the vector field $f(x, y)$ alone. The formula requires knowledge of the conjugating function h also. However, in a particular case, we give a formula such that ρ can be computed only by means of the function $f(x, y)$. Up to this time, it is not known whether ρ can be determined by the explicit expression depending on $f(x, y)$ in the ergodic case (see [**1**, §22 and **2**, p. 73]).

2. Theorems. We first prove the following

THEOREM 1. *For equation* (1), *if there is a conjugating function $h(\xi)$ with the continuous first derivative $h'(\xi)$, then the following formula holds*:

$$\rho = \int_0^1 \int_0^1 h'(\eta(x, y)) \cdot f(x, y) \cdot \exp\left(-\int_0^x f_y'(u, y)\, du\right) dx\, dy, \tag{6}$$

where $\eta(x, y)$ is designated by (5).

PROOF. Put

$$h(\eta(x, y)) = H(x, y). \tag{7}$$

The differentiability of $h(\xi)$ and the fact that $\eta(x, y)$ has continuous first partial derivatives imply that $H(x, y)$ also has continuous first partial derivatives. For any fixed point $(x_1, y_1) \in R^2$, let

$$H(x_1, y_1) = h(\eta(x_1, y_1)) = h(\eta_1),$$

and

$$H(x_1 + 1, y_1) = h(\eta(x_1 + 1, y_1)) = h(\eta_2).$$

By the expression of solutions of (1), we have

$$\varphi(x_1, \eta_1) = y_1 \quad \text{and} \quad \varphi(x_1 + 1, \eta_2) = y_1.$$

Thus,

$$\varphi(x_1, \eta_1) = \varphi(x_1 + 1, \eta_2). \tag{8}$$

From $f(x, y + 1) = f(x, y)$ of H2, it follows that

$$\varphi(x, \eta + 1) = \varphi(x, \eta) + 1. \tag{9}$$

From $f(x + 1, y) = f(x, y)$ of H2 and the fact that $\psi(\eta) = \varphi(1, \eta)$, we have

$$\varphi(x + 1, \eta) = \varphi(x, \psi(\eta)). \tag{10}$$

Setting $x = x_1$, $\eta = \eta_2$ in (10), we have $\varphi(x_1 + 1, \eta_2) = \varphi(x_1, \psi(\eta_2))$. Using (8) again, we obtain $\varphi(x_1, \eta_1) = \varphi(x_1, \psi(\eta_2))$, which together with the uniqueness of solutions imply that $\eta_1 = \psi(\eta_2)$. Applying (3) again, we obtain $h(\eta_1) = h(\eta_2) + \rho$, and $H(x_1, y_1) = H(x_1 + 1, y_1) + \rho$. Since the point (x_1, y_1) is arbitrary, it follows that

$$H(x, y) - H(x + 1, y) = \rho \tag{11}$$

for all x, y.

Further, for any fixed point $(x_1, y_1) \in R^2$, let

$$H(x_1, y_1) = h(\eta(x_1, y_1)) = h(\eta_1),$$

and

$$H(x_1, y_1 + 1) = h(\eta(x_1, y_1 + 1)) = h(\bar{\eta}_2).$$

Similarly, we have $y_1 = \varphi(x_1, \eta_1)$ and $y_1 + 1 = \varphi(x_1, \bar{\eta}_2)$. Setting $x = x_1$, $\eta = \eta_1$ in (9), we obtain

$$\varphi(x_1, \eta_1 + 1) = \varphi(x_1, \eta_1) + 1 = y_1 + 1 = \varphi(x_1, \bar{\eta}_2),$$

which together with the uniqueness of solutions of (1) imply that $\bar{\eta}_2 = \eta_1 + 1$. Using (2) again, i.e., $h(\bar{\eta}_2) = h(\eta_1 + 1) = h(\eta_1) + 1$, we obtain $H(x_1, y_1 + 1) = H(x_1, y_1) + 1$. From the arbitrariness of (x_1, y_1), this implies that

$$H(x, y + 1) = H(x, y) + 1 \tag{12}$$

for all x, y.

Put

$$\omega(x, y) = y - \rho x - H(x, y). \tag{13}$$

Using (11) and (12), we obtain

$$\begin{aligned}\omega(x + 1, y) &= y - \rho(x + 1) - H(x + 1, y) \\ &= y - \rho x - \rho + \rho - H(x, y) = \omega(x, y),\end{aligned}$$

and

$$\begin{aligned}\omega(x, y + 1) &= y + 1 - \rho x - H(x, y + 1) \\ &= y + 1 - \rho x - H(x, y) - 1 = \omega(x, y).\end{aligned}$$

Thus $\omega(x, y)$ is periodic of period one in both variables, i.e., it is a function defined on the torus. Therefore, the function $\partial H/\partial x = -\rho - \partial\omega/\partial x$ also possesses the same property.

Now, setting $x = 0$ in (11), we have

$$\rho = H(0, y) - H(1, y) = -\int_0^1 \frac{\partial H}{\partial x} dx,$$

and integrating both sides, we obtain

$$\rho = -\int_0^1 \int_0^1 \frac{\partial H(x, y)}{\partial x} dx\, dy. \tag{14}$$

Differentiating (7) with respect to x, we have

$$\frac{\partial H(x, y)}{\partial x} = h'(\eta(x, y)) \cdot \frac{\partial \eta}{\partial x}.$$

Substituting $\eta = \eta(x, y)$ into (4) and differentiating both sides with respect to x, we obtain $0 = \partial\varphi_1/\partial x + \partial\varphi_1/\partial\eta \cdot \partial\eta/\partial x$, i.e.,

$$\frac{\partial\eta}{\partial x} = -\frac{f(x, \varphi_1(x, 0, \eta(x, y)))}{\partial\varphi_1/\partial\eta}.$$

From the well-known fact that

$$\frac{\partial\varphi_1}{\partial\eta} = \exp\left(\int_0^x f_y'(x, \varphi_1(x, 0, \eta(x, y)))\, dx\right) = \exp\left(\int_0^x f_y'(x, y)\, dx\right),$$

we obtain finally

$$\frac{\partial H(x, y)}{\partial x} = h'(\eta(x, y)) \cdot \left(-f(x, y) \cdot \exp\left(-\int_0^x f_y'(u, y)\, du\right)\right).$$

Substituting this into (14), we have

$$\rho = \int_0^1 \int_0^1 h'(\eta(x, y)) \cdot f(x, y) \cdot \exp\left(-\int_0^x f_y'(u, y)\, du\right) dx\, dy.$$

This is exactly formula (6) and Theorem 1 is proved.

DEFINITION 1. If the equation (1) is ergodic with (irrational) rotation number ρ and if $\psi(\eta) = \eta + \rho$ for all $\eta \in (-\infty, +\infty)$, then (1) is said to be a rotation.

It is easy to see that, for such an equation (1), we may assume without loss of generality that $h(\xi) \equiv \xi$.

THEOREM 2. *A necessary condition that the equation* (1) *is a rotation is* $\int_0^1 f_y'(u, y)\, du = 0$. *If the equation* (1) *is a rotation, then*

$$\rho = \int_0^1 \int_0^1 f(x, y) \cdot \exp\left(-\int_0^x f_y'(u, y)\, du\right) dx\, dy. \tag{15}$$

PROOF. First, if the equation (1) is a rotation, then by virtue of $h(\xi) \equiv \xi$, substituting $h' = 1$ into (6), we obtain immediately (15).

Further, as a consequence of periodicity of $f(x, y)$, and since $\partial H(x, y)/\partial x$ is a single valued function on T^2, so the function

$$p(x, y) = \exp\left(-\int_0^x f_y'(u, y)\, du\right)$$

is also a single valued function on T^2.

Finally, it is easy to see that

$$p(x, y + 1) = p(x, y) \quad \text{and} \quad p(x + 1, y) = p(x, y) \cdot \exp\left(-\int_0^1 f_y'(u, y)\, du\right).$$

Therefore, we obtain $\exp(-\int_0^1 f_y'(u, y)\, du) = 1$, i.e., $\int_0^1 f_y'(u, y)\, du = 0$. Theorem 2 is proved.

REMARK. Consider the differential system of class C^1 on T^2 containing no singular points:

$$dx/dt = P(x, y), \qquad dy/dt = Q(x, y). \tag{16}$$

We first assume that $P(x, y) \neq 0$ for all x, y. Setting $P(x, y)\,dt = dt_1$, (16) becomes

$$(17) \qquad \frac{dx}{dt_1} = 1, \qquad \frac{dy}{dt_1} = \frac{Q(x, y)}{P(x, y)} = f(x, y).$$

If the first order equation $dy/dx = f(x, y)$ satisfies the conditions of Theorem 1 of the present paper, then the function $H(x, y)$ in (7) is exactly an integral of (17) on R^2 (since H has continuous first partial derivatives and $H =$ constant along the solutions of (17)).

We now set

$$(18) \qquad X = x, \qquad Y = \rho x + H(x, y).$$

It is clear the change of coordinates (18) transforms (17) to the following

$$(19) \qquad dX/dt_1 = 1, \qquad dY/dt_1 = \rho.$$

In the original parameter t, (19) becomes

$$(20) \qquad dX/dt = P_1(X, Y), \qquad dY/dt = \rho \cdot P_1(X, Y).$$

This means that the linearization of the trajectories of (16) has been accomplished. By virtue of (11) and (12), we have $Y(x + 1, y) = Y(x, y)$ and $Y(x, y + 1) = Y(x, y) + 1$. Hence (18) is a change of coordinates on T^2.

Analogously, if the condition $P(x, y) \neq 0$ is not satisfied, one can also make a change [3] which transforms (16) to (20) provided the first order equation $dy/dx = f(x, y)$ corresponding to (17) satisfies the conditions of Theorem 1.

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