

STABILITY OF ISOMETRIES ON BANACH SPACES

JULIAN GEVIRTZ¹

ABSTRACT. Let X and Y be Banach spaces. A mapping $f: X \rightarrow Y$ is called an ϵ -isometry if $|\|f(x_0) - f(x_1)\| - \|x_0 - x_1\|| \leq \epsilon$ for all $x_0, x_1 \in X$. It is shown that there exist constants A and B such that if $f: X \rightarrow Y$ is a surjective ϵ -isometry, then $\|f((x_0 + x_1)/2) - (f(x_0) + f(x_1))/2\| \leq A(\epsilon\|x_0 - x_1\|)^{1/2} + B\epsilon$ for all $x_0, x_1 \in X$. This, together with a result of Peter M. Gruber, is used to show that if $f: X \rightarrow Y$ is a surjective ϵ -isometry, then there exists a surjective isometry $I: X \rightarrow Y$ for which $\|f(x) - I(x)\| \leq 5\epsilon$, thus answering a question of Hyers and Ulam about the stability of isometries on Banach spaces.

Throughout, X , Y and Z denote real Banach spaces. A mapping $f: X \rightarrow Y$ is called an ϵ -isometry if $|\|f(x_0) - f(x_1)\| - \|x_0 - x_1\|| \leq \epsilon$ for all $x_0, x_1 \in X$. Hyers and Ulam [3] formulated the stability problem for isometries, that is, the question as to whether for each pair of Banach spaces X and Y there exists a constant $K = K(X, Y)$ such that for each surjective ϵ -isometry $f: X \rightarrow Y$ there exists an isometry $I: X \rightarrow Y$ for which $\|f(x) - I(x)\| \leq K\epsilon$ for all $x \in X$. This problem has been solved in a number of special cases (see [1] and [2] for a summary of such results), and Gruber [2, Theorem 1] went very far towards a general solution by showing that if $f: X \rightarrow Y$ is a surjective ϵ -isometry and $I: X \rightarrow Y$ is an isometry for which $I(0) = f(0)$ and for which $\|f(x) - I(x)\|/\|x\| \rightarrow 0$ uniformly as $\|x\| \rightarrow \infty$, then I is surjective and $\|f(x) - I(x)\| \leq 5\epsilon$ for all $x \in X$. In what follows we will show that such an isometry always exists so that the answer to the question of Hyers and Ulam is affirmative with $K(X, Y) = 5$ for all X and Y . We do this by establishing that:

There exist constants A and B such that if $f: X \rightarrow Y$ is a surjective ϵ -isometry, then

$$(1) \quad \|f((x_0 + x_1)/2) - (f(x_0) + f(x_1))/2\| \leq A(\epsilon\|x_0 - x_1\|)^{1/2} + B\epsilon$$

for all $x_0, x_1 \in X$.

(We show this with $A = 10$ and $B = 20$, but the specific values of A and B are of no consequence.)

To see that (1) indeed proves the existence of the isometry I of Gruber's result we may assume without loss of generality that $f(0) = 0$. Applying (1) with $x_0 = 2^{n+1}x$

Received by the editors December 14, 1982 and, in revised form, April 22, 1983.

1980 *Mathematics Subject Classification*. Primary 46B99.

Key words and phrases. ϵ -isometry, stability of isometries.

¹This work was supported by a grant from the Dirección de Investigación of the Universidad Católica de Chile.

and $x_1 = 0$ and dividing by 2^n we have

$$(2) \quad \|2^{-n}f(2^n x) - 2^{-n-1}f(2^{n+1}x)\| \leq 2^{-n/2}A(2\epsilon\|x\|)^{1/2} + 2^{-n}B\epsilon.$$

Since

$$f(x) - 2^{-n}f(2^n x) = \sum_{k=0}^{n-1} (2^{-k}f(2^k x) - 2^{-k-1}f(2^{k+1}x)),$$

the completeness of Y together with (2) implies that $I(x) = \lim_{n \rightarrow \infty} 2^{-n}f(2^n x)$ exists for all $x \in X$ and satisfies $I(0) = 0$ and

$$(3) \quad \|f(x) - I(x)\| \leq 2(\sqrt{2} - 1)^{-1}A(\epsilon\|x\|)^{1/2} + 2B\epsilon$$

so that $\|f(x) - I(x)\|/\|x\| \rightarrow 0$ uniformly as $\|x\| \rightarrow \infty$. Since

$$\|2^{-n}f(2^n x_0) - 2^{-n}f(2^n x_1)\| - \|x_0 - x_1\| \leq \epsilon/2^n,$$

it is clear that I is an isometry. We mention, in passing, that an observation of Gruber [2, Remark, p. 266] shows that (1) serves to eliminate some of his considerations. Indeed, (1) implies that $I((x_0 + x_1)/2) = (I(x_0) + I(x_1))/2$ so that if we assume as above that $f(0) = 0$, it follows that I is linear and in turn surjective.

To facilitate the proof of (1) we introduce some terminology. If $f: X \rightarrow Y$, then any mapping $F: Y \rightarrow X$ for which

$$(4) \quad \|fF(y) - y\| \leq \delta \quad \text{for all } y \in Y$$

is said to be a δ -inverse of f . Following Bourgin [1] f is called δ -onto if it has a δ -inverse. Henceforth the term (δ, ϵ) -isometry will refer to a δ -onto ϵ -isometry. We have:

$$(5) \quad \text{If } f: X \rightarrow Y \text{ is a } (\delta, \epsilon)\text{-isometry and } F \text{ is a } \delta\text{-inverse of } f, \text{ then } F \text{ is a } (\delta + \epsilon, 2\delta + \epsilon)\text{-isometry.}$$

To see that F is $(\delta + \epsilon)$ -onto we note that

$$\|fF(x) - x\| \leq \|fF(x) - f(x)\| + \epsilon \leq \delta + \epsilon$$

by (4). To see that F is a $(2\delta + \epsilon)$ -isometry, let $y_0, y_1 \in Y$. Then $\|fF(y_i) - y_i\| \leq \delta$ ($i = 0, 1$) by (4) and

$$\|fF(y_0) - fF(y_1)\| - \|F(y_0) - F(y_1)\| \leq \epsilon$$

since f is an ϵ -isometry. Hence

$$\|F(y_0) - F(y_1)\| - \|y_0 - y_1\| \leq \|fF(y_0) - fF(y_1)\| - \|y_0 - y_1\| + \epsilon \leq 2\delta + \epsilon.$$

We shall also need:

$$(6) \quad \text{Let } f_1: X \rightarrow Y \text{ be a } (\delta_1, \epsilon_1)\text{-isometry and let } f_2: Y \rightarrow Z \text{ be a } (\delta_2, \epsilon_2)\text{-isometry. Then } f_2 f_1 \text{ is a } (\delta_1 + \delta_2 + \epsilon_2, \epsilon_1 + \epsilon_2)\text{-isometry.}$$

It is immediate that $f_2 f_1$ is an $(\epsilon_1 + \epsilon_2)$ -isometry. To see the rest, let F_i be a δ_i -inverse of f_i ($i = 1, 2$) and let $z \in Z$. Then

$$\begin{aligned} \|f_2 f_1 F_2(z) - z\| &\leq \|f_2 f_1 F_1 F_2(z) - f_2 F_2(z)\| + \delta_2 \\ &\leq \|f_1 F_1 F_2(z) - F_2(z)\| + \epsilon_2 + \delta_2 \\ &\leq \delta_1 + \epsilon_2 + \delta_2, \end{aligned}$$

where we have used (4) as it applies to f_2 and f_1 and the fact that f_2 is an ε_2 -isometry.

The proof of (1) which we now present is an adaptation of a proof given by Vogt [4] of a generalization of the Mazur-Ulam theorem on isometries. Let $x_0, x_1 \in X$, $y_i = f(x_i)$ ($i = 0, 1$), $p = (x_0 + x_1)/2$ and $q = (y_0 + y_1)/2$. Until the very end of the proof we shall assume that $y_0 \neq y_1$. Since f is a $(0, \varepsilon)$ -isometry, by (5) it has a 0-inverse F which is an $(\varepsilon, \varepsilon)$ -isometry and for which $F(y_i) = x_i$ ($i = 0, 1$). We define sequences $(g_k)_{k \geq 0}$ and $(G_k)_{k \geq 0}$ of mappings of Y into Y with the following properties:

$$(7) \quad g_k \text{ is a } (4^{k+1}\varepsilon, 4^{k+1}\varepsilon)\text{-isometry and } g_k(y_i) = y_{1-i} \quad (i = 0, 1),$$

$$(8) \quad G_k \text{ is a } 4^{k+1}\varepsilon\text{-inverse of } g_k \text{ and } G_k(y_i) = y_{1-i} \quad (i = 0, 1).$$

To begin we let $g_0(y) = f(2p - F(y))$ for $y \in Y$. By (6), g_0 is a $(2\varepsilon, 2\varepsilon)$ -isometry and it is clear that it permutes y_0 and y_1 . Thus (7) holds for $k = 0$. We let G_0 be any mapping which satisfies (8) for $k = 0$. Next, we let $g_1(y) = G_1(y) = 2q - y$ for $y \in Y$. Obviously (7) and (8) are then satisfied for $k = 1$. Finally, assuming that we have g_0, \dots, g_n and G_0, \dots, G_n which satisfy the stipulated conditions, we define $g_{n+1} = g_{n-1}g_nG_{n-1}$. A simple argument based on (5) and (6) shows that g_{n+1} satisfies (7) with $k = n + 1$. G_{n+1} is then taken to be any mapping satisfying (8) with $k = n + 1$.

We next define a sequence $(a_n)_{n \geq 1}$ of points of Y recursively by $a_1 = q$ and $a_{n+1} = g_{n-1}(a_n)$ for $n \geq 1$. Let $d = \|y_0 - y_1\|/2$. Denoting by $B(y, r)$ the closed ball of radius r and center y , we have that $g_k(B(y_i, r)) \subset B(y_{1-i}, r + 4^{k+1}\varepsilon)$. Since $a_1 \in B(y_0, d) \cap B(y_1, d)$ and $a_n = g_{n-2}g_{n-3} \cdots g_0(a_1)$, successive application of this inclusion with $k = 0, 1, \dots, n - 2$ yields

$$a_n \in B(y_0, d + 4^n\varepsilon) \cap B(y_1, d + 4^n\varepsilon) \subset B(q, d + 4^n\varepsilon).$$

Since the diameter of this last ball is $2(d + 4^n\varepsilon)$ we conclude that

$$(9) \quad \|a_n - a_{n-1}\| \leq 2(d + 4^n\varepsilon) \quad \text{for } n \geq 2.$$

We now show that for all $y \in Y$ there holds

$$(10) \quad \|g_n(y) - y\| \geq 2\|a_n - y\| - 2(4^n - 1)\varepsilon \quad \text{for } n \geq 1.$$

Since $g_1(y) = 2q - y$, this is true for $n = 1$. Assuming that it is valid for a given $n \geq 1$, we have

$$\begin{aligned} \|g_{n+1}(y) - y\| &= \|g_{n-1}g_nG_{n-1}(y) - y\| \\ &\geq \|g_{n-1}g_nG_{n-1}(y) - g_{n-1}G_{n-1}(y)\| - 4^n\varepsilon \\ &\geq \|g_nG_{n-1}(y) - G_{n-1}(y)\| - 2 \cdot 4^n\varepsilon \\ &\geq 2\|a_n - G_{n-1}(y)\| - (4^{n+1} - 2)\varepsilon \\ &\geq 2(\|g_{n-1}(a_n) - g_{n-1}G_{n-1}(y)\| - 4^n\varepsilon) - (4^{n+1} - 2)\varepsilon \\ &\geq 2(\|a_{n+1} - y\| - 2 \cdot 4^n\varepsilon) - (4^{n+1} - 2)\varepsilon \\ &= 2\|a_{n+1} - y\| - 2(4^{n+1} - 1)\varepsilon, \end{aligned}$$

so that (10) holds for all $n \geq 1$ by induction. (Here we have used in order: the definition of g_{n+1} , (4) as applied to g_{n-1} with $\delta = 4^n\varepsilon$, the fact that g_{n-1} is a

$4^n\epsilon$ -isometry, the inductive hypothesis, the fact that g_{n-1} is a $4^n\epsilon$ -isometry once again, and finally the definition of a_{n+1} together with (4) as applied to g_{n-1} .) The bound (10) implies that $\|a_{n+1} - a_n\| = \|g_{n-1}(a_n) - a_n\| \geq 2\|a_n - a_{n-1}\| - 2 \cdot 4^{n-1}\epsilon$, which by induction gives

$$\|a_n - a_{n-1}\| \geq 2^{n-2}\|a_2 - a_1\| - 4^{n-1}\epsilon.$$

Together with (9) this means that, for $n \geq 2$, $\|a_2 - a_1\|$ is bounded above by $2^{2-n}(2d + 2 \cdot 4^n\epsilon + 4^{n-1}\epsilon)$ or, equivalently,

$$(11) \quad \|a_2 - a_1\| \leq 2(d2^{-n} + 18\epsilon 2^n) \quad \text{for } n \geq 0.$$

We have

$$\begin{aligned} \|a_2 - a_1\| &= \|f(2p - F(q)) - q\| = \|f(2p - F(q)) - fF(q)\| \\ &\geq 2\|p - F(q)\| - \epsilon \geq 2(\|f(p) - fF(q)\| - \epsilon) - \epsilon \\ &= 2\|f(p) - q\| - 3\epsilon, \end{aligned}$$

so that by (11)

$$\|f(p) - q\| \leq d2^{-n} + 18\epsilon 2^n + 2\epsilon \quad \text{for } n \geq 0.$$

For the moment we assume that $d > 18\epsilon$ and let t be such that $d2^{-t} = 18\epsilon 2^t$; that is, $t = (\log 4)^{-1} \log(d/18\epsilon) > 0$. If we let n be the greatest integer less than or equal to t , the above bound for $\|f(p) - q\|$ gives

$$\begin{aligned} \|f(p) - q\| &\leq 2d2^{-t} + 18\epsilon 2^t + 2\epsilon = 3d2^{-t} + 2\epsilon \\ &= 3(18\epsilon d)^{1/2} + 2\epsilon \leq 10(\epsilon\|x_0 - x_1\|)^{1/2} + 2\epsilon, \end{aligned}$$

since $\|x_0 - x_1\| \geq \|y_0 - y_1\| - \epsilon = 2d - \epsilon \geq 35d/18$. On the other hand, if $d \leq 18\epsilon$ (which covers the case $y_0 = y_1$ that was excluded at the beginning of the proof), then $\|y_0 - y_1\| \leq 36\epsilon$ and so $\|x_0 - x_1\| \leq 37\epsilon$. Thus $\|x_i - p\| \leq 19\epsilon$ and, consequently, $\|y_i - f(p)\| \leq 20\epsilon$ ($i = 0, 1$). Since $q = (y_0 + y_1)/2$ we have $\|f(p) - q\| \leq 20\epsilon$. Therefore in either case there holds $\|f(p) - q\| \leq 10(\epsilon\|x_0 - x_1\|)^{1/2} + 20\epsilon$.

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FACULTAD DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, CASILLA 114-D, SANTIAGO, CHILE