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# MINIMAX AND VARIATIONAL INEQUALITIES FOR COMPACT SPACES

J. F. McCLENDON

ABSTRACT. The minimax inequality $\min_y \sup_x f(x, y) \leq \sup_x f(x, x)$, proved by K. Fan for convex spaces, is proved here for certain contractible and acyclic spaces. Some variational inequality and fixed point theorems are deduced.

In [**10**] Fan proved a minimax inequality for $f\colon X \times X \to R$, namely,

$$\min_y \sup_x f(x, y) \leq \sup_x f(x, x).$$

Fan assumed $X$ was a compact convex subset of a Hausdorff topological vector space $E$ and, also, made some assumptions about $f$. In the present paper the inequality is proved under different assumptions on $X$ and $f$. The main generalization is the weakening of "convex" to "contractible" or "acyclic". Because of other changes neither of the minimax inequality results (§2) strictly generalizes Fan's result. Theorem 2.1 does include Fan's result if the topological vector space $E$ is finite dimensional. In the infinite-dimensional case, Theorem 2.2 includes and generalizes compact convex subsets of a locally convex topological vector space, but imposes more restrictions on $f$. In §3 some variational inequality (see, for example, Browder [**6**], Brezis-Nirenberg-Stampacchia [**4**], Dugundji-Granas [**8**] for convex analogues) and fixed point theorems are deduced from the minimax inequality.

**1. Terminology, background results.** The real numbers are $R$. A space is *contractible* if the identity map is homotopic to a constant (so a contractible space is always nonempty). A nonempty convex set is contractible—but, of course, there are many contractible sets in a topological vector space which are not convex. A nonempty space is *acyclic* if it is connected and its Čech homology (coefficients in a fixed field) is zero in dimensions greater than zero. Every contractible space is acyclic and there are examples showing the converse is not true (e.g., Brown [**7**, p. 31]).

A set valued function $m\colon X \to Y$ with $m(x)$ nonempty for all $x$ is called a *multifunction*. The graph $G(m)$ is $\{(x, y) \in X \times Y \mid y \in m(x)\}$. $m$ is said to be an open-graph [closed-graph] multifunction if $G(m)$ is open [closed] in $X \times Y$. If $m\colon X \to X$ then $x \in X$ is a fixed point of $m$ if $x \in m(x)$. We will use two fixed point results: Proposition 2.3 of §2 and the following result of Begle.

1.1. PROPOSITION (BEGLE [**2**]). *Suppose $X$ is a compact acyclic lc space and $m\colon X \to X$ is a closed-graph acyclic valued multifunction. Then $m$ has a fixed point.*

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1.2. PROPOSITION. (a) (*Lefschetz*) *An ANR is an lc space.* (b) (*R. Thompson* **[14]**) *A finite union of compact convex subsets of a locally convex topological vector space is an lc space.*

See [**2**, **14**] for the definition of "lc space". It will be convenient to define an *fc space* to be a finite union of compact convex subsets of a locally convex topological vector space. The following lemma is needed. Let $f: X \to R$ be an upper semicontinuous function and let

$$V(t) = \{x \in X \mid f(x) > t\} \quad \text{and} \quad H(t) = \{x \in X \mid f(x) \geqslant t\}.$$

1.3. LEMMA. *If $X$ is compact and $V(t)$ is acyclic for all $t$ then $H(t)$ is acyclic for all $t$.*

PROOF. $H(t) = \bigcap\{V(t-\varepsilon) \mid \varepsilon > 0\}$. Using $X$ compact and $f$ usc, it is easily checked that $H(t) \subset$ open $W$ implies $H(t) \subset V(t-\varepsilon) \subset W$, some $\varepsilon > 0$. Now the result follows from the continuity property of Čech homology.

ANR will always mean ANR (metric). The reader may find it helpful to remember that if $X$ is a subset of Euclidean space then $X$ is a compact finite-dimensional ANR iff $X$ is compact and locally contractible (see Borsuk [**3**, p. 122]).

Henceforth $X$ will always be a *nonempty* space.

**2. Minimax inequality results.** In this section we consider $f: X \times X \to R$, with $\sigma = \sup\{f(x, x) \mid x \in X\}$. The following two theorems will be proved.

2.1. THEOREM. *Suppose that $X$ is a compact acyclic finite-dimensional ANR. Suppose also that*:

(1) $\{(x, y) \in X \times X \mid f(x, y) > \sigma\}$ *is open.*

(2) $\{x \in X \mid f(x, y) > \sigma\}$ *is contractible or empty for all $y \in X$.*

*Then there is a $y_0 \in X$ with $f(x, y_0) \leqslant \sigma$, all $x \in X$.*

The next theorem relaxes the hypotheses on $X$ but strengthens those on $f$.

2.2. THEOREM. *Suppose that $X$ is compact, acyclic, and either an fc space or an ANR. Suppose also*

(1) $\{(x, y) \mid f(x, y) < t\}$ *open, all $t \in R$,*

(2) $\{y \mid f(x, y) > t\}$ *open, all $x \in X$, all $t \in R$,*

(3) $\{x \mid f(x, y) > t\}$ *acyclic or empty, all $y \in Y$, all $t \in R$.*

*Then* $\min_y \sup_x f(x, y) \leqslant \sigma$.

Note that if $f$ is lower semicontinuous then 2.1(1) is satisfied and, also, the conclusion of 2.1 is then equivalent to: $\min_y \sup_x f(x, y) \leqslant \sigma$. As stated, 2.2(1) says $f$ is upper semicontinuous and 2.2(2) says $f(x, -)$ is lower semicontinuous. However, the proof of 2.2 will show that it suffices to take $t = \sigma + \varepsilon$ for $\varepsilon$ small and positive. It will also show that $>$ can be replaced by $\geqslant$ in (3). As noted in the introduction, Theorem 2.1 includes the finite-dimensional version of Fan's result [**10**].

PROOF OF 2.1. Define $m: X \to X$ by $m(y) = \{x \mid f(x, y) > \sigma\}$. Hypothesis (1) shows that $m$ is an open-graph multifunction. The following fixed point theorem will be needed.

2.3. PROPOSITION [**11**, COROLLARY 3.3]. *Suppose $X$ is a compact acyclic finite-dimensional ANR and $m$: $X \to X$ a subopen multifunction with infinitely connected values. Then $m$ has a fixed point.*

By hypothesis (2), $m$ has contractible so infinitely connected values. We have "subopen" is a generalization of open-graph. If $m$ is a multifunction, then the above proposition will yield a fixed point for $m$. Then there is a point $y'$ with $y' \in m(y')$. But this says that $f(y', y') > \sigma = \sup_x f(x, x)$ which is not possible. Hence $m$ is not a multifunction and there is a point $y_0$ so that $m(y_0)$ is empty, i.e., $f(x, y_0) \leqslant \sigma$ for all $x \in X$, proving 2.1.

In [**13**] it is proved that "subopen" can be replaced by (for example) "$r$-open graph" in 2.3 above. Thus in 2.1, if $L = \{(x, y) | f(x, y) > \sigma\}$ hypothesis (1) can be replaced by the following: (1′) There is an open set $U$ of $X \times X$ and a continuous retraction $r = (r_1, r_2)$: $U \to L$ such that $r_2(a, b) = b$ for all $b \in X$.

PROOF OF 2.2. We can assume $\sigma < \infty$ and define for each $\varepsilon > 0$, $m(y) = \{x | f(x, y) \geqslant \sigma + \varepsilon\}$. By hypothesis (1) $m$ is a closed-graph multifunction and (2) says that $f(x, -)$ is lower semicontinuous. (3) and Lemma 1.3 show that $m$ has acyclic values. Since $m$ cannot have a fixed point, Propositions 1.1 and 1.2 show that $m(y') = \varnothing$ for some $y' = y'(\varepsilon)$. By taking $\varepsilon = 1/n$ we get a sequence and then a convergent subnet ($X$ compact) $y(k) \to \bar{y}$ with $f(x, y(k)) \leqslant \sigma + \varepsilon(k)$, $\varepsilon(k) \to 0$. However, $f(x, -)$ is lower semicontinuous so $f(x, \bar{y}) \leqslant \sigma$. Thus $\sup_x f(x, \bar{y}) \leqslant \sigma$ and $\inf_y \sup_x f(x, y) \leqslant \sigma$ and because $X$ is compact and $f(x, -)$ is lower semicontinuous "inf" can be replaced by "min".

**3. Variational inequality and fixed point theorems.** We first deduce two preliminary results, along the lines of Fan [**10**, Corollary 1].

3.1. THEOREM. *Suppose that $X$ is a compact acyclic finite-dimensional ANR. Suppose $g$: $X \times X \to R$ is a function such that $\{(x, y) | g(y, y) > g(x, y)\}$ is open and $\{x | g(y, y) > g(x, y)\}$ is contractible or empty for all $y \in X$. Then there is a $y_0 \in X$ with $g(y_0, y_0) \leqslant g(x, y_0)$ for all $x \in X$.*

PROOF. Define $f$: $X \times X \to R$, $f(x, y) = g(y, y) - g(x, y)$. Then $\sigma = \sup_x f(x, x) = 0$ so $f(x, y) > \sigma$ iff $g(y, y) > g(x, y)$ and hypotheses (1) and (2) of Theorem 1.2 are satisfied. Thus there is a $y_0 \in X$ with $f(x, y_0) \leqslant 0$, all $x \in X$, proving 3.1.

3.2. THEOREM. *Suppose that $X$ is compact, acyclic, and either an fc space or an ANR. Let $g$: $X \times X \to R$ be a function such that* (a) *$g$ is continuous on the diagonal* ($= \{(x, x) | x \in X\}$), (b) *$g$ is lower semicontinuous, and* (c) *$g(x, -)$ is upper semicontinuous for all $x \in X$. Suppose also that $\{x | g(x, y) < s\}$ is acyclic or empty for all $s \in R$, all $y \in X$. Then there is a $y_0 \in X$ with $g(y_0, y_0) \leqslant g(x, y_0)$, all $x \in X$.*

PROOF. Define $f$: $X \times X \to R$, $f(x, y) = g(y, y) - g(x, y)$. The hypotheses on $g$ imply that $f$ is upper semicontinuous and that $f(x, -)$ is lower semicontinuous. Thus (1) and (2) of Theorem 1.2 are satisfied. For a given $y$ and $t$ let $s = g(y, y) - t$. Then (3) of 2.1 follows from the above hypotheses on $g$. The conclusion of 3.2 then follows from that of 2.1.

There are a large number of variational inequality results in the literature using convexity. See, for example, Browder [**5**, **6**], Brezis-Nirenberg-Stampacchia, [**4**], Allen [**1**], Dugundji-Granas [**8**]. Here we state some results for acyclic spaces.

3.3. COROLLARY. *Let $E$ be a topological vector space with algebraic dual $E^*$. Suppose $E \supset X$ and $X$ is a compact acyclic finite-dimensional ANR. Suppose $T$: $X \to E^*$ is a function such that*

(1) $\{(x, y) \in X \times X \mid (Ty, x) < (Ty, y)\}$ *is open*,

(2) $\{x \in X \mid (Ty, x) < (Ty, y)\}$ *is contractible or empty, all $y \in X$. Then there is a $y_0 \in X$ with $(Ty_0, y_0 - x) \leq 0$, all $x \in X$.*

3.4. COROLLARY. *Let $E$ be a topological vector space with algebraic dual $E^*$. Suppose $E \supset X$ and $X$ is compact, acyclic, and either fc or ANR. Suppose $T$: $X \to E^*$ is a function such that $y \to (Ty, y)$ is continuous, $y \to (Ty, x)$ is upper semicontinuous, all $x$, and $(x, y) \to (Ty, x)$ is lower semicontinuous. Suppose $\{x \in X \mid (Ty, x) < s\}$ is acyclic or empty, all $s \in R$, all $y \in X$. Then there is a $y_0 \in X$ with $(Ty_0, y_0 - x) \leq 0$ for all $x \in X$.*

3.3 and 3.4 are proved by using $g(x, y) = (Ty, x)$ in 3.1 and 3.2. A version of 3.3 that allows $X$ to be noncompact (but still finite dimensional) will be proved after the fixed point theorems below. These fixed point theorems treat a function $f$: $X \to M$ where $X \subset M$ and involve a condition on the "boundary". There are many theorems of this general type in the literature for convex $X$ (see, e.g., Browder [**5**, **6**]).

3.5. DEFINITION. Suppose $X$ a subset of a metric space $(M, d)$. Define $\delta X = \delta_M X = \{y \in X \mid \text{There is a } z \in M \setminus X \text{ with } d(z, y) = d(z, X)\}$.

3.6. THEOREM. *Suppose $(M, d)$ is a metric space, $M \supset X$ and $X$ a compact, acyclic, finite-dimensional ANR. Suppose $f$: $X \to M$ is a continuous function with $f(\delta X) \subset X$ and $\{x \in X \mid d(y, f(y)) > d(x, f(y))\}$ contractible or empty for all $y \in X$. Then $f$ has a fixed point.*

PROOF. Define $g$: $X \times X \to R$ by $g(x, y) = d(x, fy)$. Then 3.1 gives $y_0 \in X$ with $d(y_0, fy_0) \leq d(x, fy_0)$ for all $x \in X$. Suppose, by way of contradiction, that $f(y_0) \in M \setminus X$. Use $z = f(y_0)$ in the definition of $\delta X$ and conclude $y_0 \in \delta X$. But by hypothesis, $f(\delta X) \subset X$. Thus we must have $f(y_0) \in X$. Now use $x = f(y_0)$ on the right-hand side of the above inequality and conclude $d(y_0, fy_0) = 0$. So $y_0$ is the desired fixed point.

3.7. THEOREM. *Suppose $(M, d)$ a metric space, $X$ a compact acyclic lc subspace of $M$. Suppose $f$: $X \to M$ is a continuous function such that $\{x \in X \mid d(x, fy) \leq t\}$ is acyclic or empty for all $y \in X$ and $t \in R$. Then there is a $y_0 \in X$ with $d(fy_0, y_0) = d(fy_0, X)$.*

The proof of 3.7 is just the first part of that of 3.6 but using 3.2. Note that 3.7 generalizes Theorem 2 of Fan [**9**].

3.8. COROLLARY. *Suppose $M$, $X$, $f$ as in 3.7 and $f(\delta X) \subset X$. Then $f$ has a fixed point.*

PROOF. Suppose, by way of contradiction, the $y_0$ of 3.7 is in $M \setminus X$. Then using $z = fy_0$ in 3.5 we see that $y_0$ is in $\delta X$ showing $fy_0 \in X$. Now for $fy_0 \in X$ the conclusion of 3.7 gives $d(fy_0, y_0) = 0$ so $y_0$ is the desired fixed point.

We conclude with another variational inequality result. The condition ($\star$) of the result seems a littel severe (cf. the weaker condition in Brezis-Nirenberg-Stampacchia **[4]**) but the methods of proof used here seem to require it.

3.9. DEFINITION. Let $S$ be a subset of a metric space $M$. Say $S$ is *simple* if $S \cap B(z; \varepsilon)$ is contractible or empty for all $z \in M$, all $\varepsilon > 0$.

3.10. THEOREM. *Let $E$ be a topological vector space and $X$ any metric subset. Suppose $T$: $X \to E^*$ is a function such that $\{(x, y) \in X \times X \mid (Ty, x) < (Ty, y)\}$ is open and $\{x \in X \mid (Ty, x) < (Ty, y)\}$ is contractible or empty for all $y \in X$. Suppose there is a simple $K \subset X$ that is an acyclic compact finite-dimensional ANR satisfying*

$$(\star) \qquad x \in X \setminus K \Rightarrow (Ty, y - x) \leqslant 0, \quad \text{for all } y \in \delta_X(K).$$

*Then there is a $y_0 \in X$ with $(Ty_0, y_0 - x) \leqslant 0$, all $x \in X$.*

PROOF. Define $m$: $K \to X$ by $m(y) = \{x \in X \mid (Ty, x) < (Ty, y)\}$. By hypothesis $m$ is open-graph and has contractible or empty values. Assume, by way of contradiction, that $m$ has nonempty values. Then, by **[12**, Corollary 2.2], $m$ has a continuous selection $f$: $K \to X$. Let $y \in \delta K$, then $fy \in my$ so $(Ty, fy) < (Ty, y)$. Then condition ($\star$) with $x = fy$ shows we must have $fy \in K$. Thus $f(\delta K) \subset K$. Also $\{x \in K \mid d(x, fy) < d(y, fy)\}$ is $B(fy; \varepsilon) \cap K$ where $\varepsilon = d(y, fy)$ and this is contractible or empty since $K$ is simple. The hypotheses of 3.6 are satisfied, giving a $y' = fy' \in my'$. But this is not possible—so $my_0$ must be empty for some $y_0 \in K$. Thus $(Ty_0, y_0) \leqslant (Ty_0, x)$ for all $x \in X$, proving 3.10.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66045