

TARSKI'S EXTENSION THEOREM FOR GROUP-VALUED CHARGES

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ABSTRACT. A result of Tarski on extensions of real-valued charges is extended to group-valued charges for certain groups.

1. Introduction and notation. A finitely additive function on a field of sets taking values in a group will be called a charge. The purpose of this paper is to examine and extend the validity of the following theorem of Tarski (see [3 and 1]) for group-valued charges. 2^X is the power set of a set X .

TARSKI'S THEOREM. *If \mathcal{C} is a field of subsets of a set X , then any real-valued charge on \mathcal{C} can be extended as a real-valued charge on 2^X .*

We assume all groups are commutative, though this is unnecessary. For any collection \mathcal{F} of subsets of X , $\langle \mathcal{F} \rangle$ is the field generated by \mathcal{F} . $|A|$ is the cardinality of a set A .

2. Results. We start with an elementary lemma.

LEMMA 1. *Let \mathcal{C} be a finite field of subsets of X , G a group, μ a G -valued charge on \mathcal{C} , and $A \notin \mathcal{C}$. Then there is a G -valued charge on $\langle \mathcal{C}, A \rangle$ which is an extension of μ .*

PROOF. We shall exhibit a charge τ on $\mathcal{P}(X)$ which extends μ . Since \mathcal{C} is a finite field there is a partition A_1, A_2, \dots, A_k of X consisting of nonempty sets from \mathcal{C} such that \mathcal{C} is the collection of all possible unions of sets from this partition. Fix points $x_i \in A_i$ for $i = 1, \dots, k$. For any $A \subset X$, if we define $\tau(A) = \sum_{x_i \in A} \mu(A_i)$, then τ is a G -valued charge on $\mathcal{P}(X)$ which extends μ .

THEOREM 2. *Let G be a compact T_2 topological group. Let μ be a G -valued charge defined on a field \mathcal{C} of subsets of a set X . Then μ can be extended to a G -valued charge on 2^X .*

PROOF. Let $A \notin \mathcal{C}$. We shall show that μ can be extended as a G -valued charge to $\langle \mathcal{C}, A \rangle$. Then the extension to 2^X follows by the usual transfinite argument. Let $\mathcal{B} = \langle \mathcal{C}, A \rangle$.

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Let $G^{\mathfrak{B}}$ be the set of all functions defined on \mathfrak{B} taking values in G and equipped with the product topology. Then $G^{\mathfrak{B}}$ is a compact T_2 space. For any field $\mathfrak{D} \subset \mathfrak{B}$, let $L_{\mathfrak{D}}$ be the set of all elements of $G^{\mathfrak{B}}$ which are additive on \mathfrak{D} , i.e.,

$$L_{\mathfrak{D}} = \bigcap_{\substack{C, D \in \mathfrak{D} \\ C \cap D = \emptyset}} \{ \tau \in G^{\mathfrak{B}} : \tau(C \cup D) = \tau(C) + \tau(D) \}.$$

Then clearly $L_{\mathfrak{D}}$ is a closed subset of $G^{\mathfrak{B}}$.

For any finite field $\mathfrak{F} \subset \mathcal{C}$ if we define

$$M_{\mathfrak{F}} = \{ \tau \in L_{\langle \mathfrak{F}, A \rangle} : \tau(B) = \mu(B) \text{ for all } B \in \mathfrak{F} \}$$

then $M_{\mathfrak{F}}$ is a closed subset of $G^{\mathfrak{B}}$ and is nonempty by Lemma 1. Also the family $\{M_{\mathfrak{F}} : \mathfrak{F} \text{ is a finite subfield of } \mathcal{C}\}$ has the finite intersection property. Since $G^{\mathfrak{B}}$ is compact, there is a $\tau \in M_{\mathfrak{F}} \forall \mathfrak{F} \subset \mathcal{C}$ which is a G -valued charge on \mathfrak{B} extending μ because $\bigcup \langle \mathfrak{F}, A \rangle$, the union over all \mathfrak{F} , equals \mathfrak{B} .

REMARK 3. Theorem 2 is also valid for any algebraically compact group (see [2] for the definition and properties). If G is algebraically compact there is a compact T_2 topological group H containing G and K , a subgroup of H , such that $G \oplus K = H$. Considering μ as an H -valued charge, by the previous case, we extend μ to an H -valued charge τ on 2^X . Now, if we compose τ with the projection to G from H , we get a G -valued charge which is an extension of μ . Since every divisible group is algebraically compact, Theorem 2 is also valid for any divisible group, a fact which can also be proved otherwise.

REMARK 4. If μ is a charge on a field of sets \mathfrak{B} taking values in a compact T_2 topological group, then μ need not be exhaustive (i.e. $\mu(A_n)$ need not converge to 0 for every pairwise disjoint sequence of sets $A_n, n \geq 1$, from \mathfrak{B}). Let $G = \{0, 1\}^{\mathbb{N}_0}$ be the countable product of the discrete two-point topological group $\{0, 1\}$. Let g_1, g_2, \dots be a countable dense subset of G . Let \mathfrak{B} be the finite cofinite field on the set of positive integers. Define μ on \mathfrak{B} by

$$\begin{aligned} \mu(B) &= \sum_{i \in B} g_i \quad \text{if } B \text{ is finite,} \\ &= \sum_{i \notin B} g_i \quad \text{if } B \text{ is cofinite.} \end{aligned}$$

Then μ is a charge on \mathfrak{B} taking values in a compact T_2 group which is not exhaustive. Thus Theorem 2 does not follow from the many extension theorems which followed [5] (see for example Theorem 3 of [4]).

REMARK 5. Since any finite group is a compact T_2 group, Theorem 2 holds for any finite group. If G is the two-element group $\{0, 1\}$, or more generally, if every element of G is of order 2, we can prove Theorem 2 without resorting to topological arguments. This is so because a function μ on a field \mathcal{C} into G is a charge if and only if $\mu(A \Delta B) = \mu(A) + \mu(B) \forall A, B \in \mathcal{C}$. Hence, a function μ on a field \mathcal{C} into G is a charge if and only if it is a group homomorphism from \mathcal{C} considered as a group with Δ .

As a final result, we shall generalize Lemma 1 by replacing the finiteness of \mathcal{C} with the finiteness of the range of μ .

THEOREM 6. *Let G be a group. Let μ be a G -valued charge on a field \mathcal{C} of subsets of X whose range is a finite set. Then μ can be extended as a G -valued charge to any field \mathfrak{B} containing \mathcal{C} .*

PROOF. Let $\mu(\mathcal{C})$ be the range of μ on \mathcal{C} . Let H be the subgroup of all elements of G of finite order.

Case (i). $\mu(\mathcal{C}) \subset H$. In this case the subgroup K generated by $\mu(\mathcal{C})$ is finite and μ can be considered as a K -valued charge on \mathcal{C} , and by Theorem 3 there is a K -valued charge on \mathfrak{B} extending μ .

Case (ii). $\mu(\mathcal{C}) \not\subset H$, i.e., $\mu(\mathcal{C}) - H \neq \emptyset$. Let

$$\mathfrak{P} = \{ \{A_1, \dots, A_n\} : A_i \neq \emptyset, A_i \in \mathcal{C}, \mu(A_i) \notin H \\ \text{for } i = 1, \dots, n \text{ and } A_i \cap A_j = \emptyset \text{ for } i \neq j \}.$$

Then if $\{A_1, \dots, A_n\} \in \mathfrak{P}$, for any fixed $a \notin H$, $|\{i : \mu(A_i) = a\}| \leq |\mu(\mathcal{C})|$, so $n \leq |\mu(\mathcal{C})|^2$. Let $\{A_1, \dots, A_p\} \in \mathfrak{P}$ be such that p is the maximum possible n for collections $\{A_1, \dots, A_n\}$ in \mathfrak{P} . We shall show that $\mu|_{A_i \cap \mathcal{C}}$ can be extended to $A_i \cap \mathfrak{B}$ as a G -valued charge.

Let us concentrate on A_1 . Consider

$$\mathfrak{F} = \{ B \subset A_1 : B \in \mathfrak{B} \text{ and } \mu(B) \notin H \}.$$

Then (i) $A_1 \in \mathfrak{F}$; (ii) $B \in \mathfrak{F}, B \subset C, C \in \mathcal{C} \Rightarrow C \in \mathfrak{F}$ because if $\mu(C) \in H$ then $\mu(A_1 - C) \notin H$ and $\mu(C - B) \notin H$, which contradicts the choice of $\{A_1, \dots, A_p\}$; (iii) $B \in \mathfrak{F}, C \in \mathfrak{F} \Rightarrow B \cap C \in \mathfrak{F}$ because if $\mu(B \cap C) \in H$ then $\mu(B - B \cap C) \notin H$ and $\mu(C - B \cap C) \notin H$ and $\{B - B \cap C, C - B \cap C, A_2, \dots, A_p\}$ gives another element of \mathfrak{P} containing $p + 1$ elements; (iv) $B \subset A_1, B \in \mathcal{C} \Rightarrow B \in \mathfrak{F}$ or $A_1 - B \in \mathfrak{F}$ and exactly one of them $\in \mathfrak{F}$; and (v) $\emptyset \notin \mathfrak{F}$.

Thus \mathfrak{F} is a maximal filter in the field $A_1 \cap \mathcal{C}$ on A_1 . Let us define μ' on $A_1 \cap \mathcal{C}$ by

$$\begin{aligned} \mu'(C) &= \mu(A_1) & \text{if } C \subset A_1, C \in \mathfrak{F}, \\ &= 0 & \text{if } C \subset A_1, C \in \mathcal{C} \text{ and } C \notin \mathfrak{F}. \end{aligned}$$

Then μ' is a charge on $A_1 \cap \mathcal{C}$ and for any $A \in \mathcal{C}, A \subset A_1$, $\mu(A) - \mu'(A)$ equals $\mu(A - A_1)$ or $\mu(A)$ according as $\mu(A) \notin H$ or not. By (iv) this implies $\mu(A) - \mu'(A) \in H$ for all $A \in \mathcal{C}, A \subset A_1$, or $\mu - \mu'$ is an H -valued charge on $A_1 \cap \mathcal{C}$ with finite range. Hence by Case (i), $\mu - \mu'$ can be extended to an H -valued charge τ_1 on $A_1 \cap \mathfrak{B}$. On the other hand, by extending the ultrafilter \mathfrak{F} in $A_1 \cap \mathcal{C}$ to an ultrafilter \mathfrak{F}' in $A_1 \cap \mathfrak{B}$ and by defining τ_2 on $A_1 \cap \mathfrak{B}$ by $\tau_2(A) = \mu(A_1)$ if $A \in \mathfrak{F}'$ and $\tau_2(A) = 0$ if $A \notin \mathfrak{F}'$, $A \in A_1 \cap \mathfrak{B}$, we obtain a charge on $A_1 \cap \mathfrak{B}$ extending μ' . Now $\tau_1 + \tau_2$ is a G -valued charge on $A_1 \cap \mathfrak{B}$ extending $\mu|_{A_1 \cap \mathcal{C}}$.

Since this procedure can be adopted to each of the A_i 's, we obtain our result.

REMARK 6. The extension we have obtained in Theorem 6 also has finite range.

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