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# THE GELFAND SUBALGEBRA OF REAL OR NONARCHIMEDEAN VALUED CONTINUOUS FUNCTIONS

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ABSTRACT. Let $L$ be either the field of real numbers or a nonarchimedean rank-one valued field. For topological space $T$ we study the Gelfand subalgebra $C_0(T, L)$ of the algebra of all $L$-valued continuous functions $C(T, L)$. The main result is that if $T$ is a paracompact locally compact Hausdorff space, which is ultraregular if $L$ is nonarchimedean, then for $f \in C(T, L)$ the following statements are equivalent: (1) There exists a compact set $K \subset T$ such that $f(T - K)$ is finite, (2) $f$ has finite range on every discrete closed subset of $T$, and (3) $f \in C_0(T, L)$.

Throughout this paper $L$ will stand for either the valued field $\mathbf{R}$ or a nonarchimedean rank-one valued field, and $T$ for a Hausdorff topological space. $T$ will be assumed completely regular in the real case and ultraregular in the nonarchimedean case.

We will denote by $C(T, L)$, or simply by $C$ if there is no confusion, the algebra over $L$ consisting of all $L$-valued continuous functions on $T$, and by $C_K$ the ideal of those continuous $f \in C$ with compact support. Let $\mathfrak{M}$ be the set of maximal ideals of $C$, and denote by $C_0$ the Gelfand subalgebra of $C$, consisting of all $f \in C$ with the property that, for each $M \in \mathfrak{M}$, there exists $\lambda \in L$ such that $(f - \lambda) \in M$. (The concept of Gelfand subalgebras of general algebras has been introduced by N. Shell —formerly N. Shilkret—in **[6]**.) We will denote by $C_F$ the subalgebra of $C$ consisting of those $f \in C$ for which there exists a compact set $K \subset T$ such that $f(T - K)$ is finite.

PROPOSITION 1. *For any* $f \in C_0$, $f(T)$ *is compact.*

PROOF. First, we prove that $f(T)$ is a precompact set.

*Real case*. It suffices to see that $f$ is bounded, and this follows from **[5**, 5.7(b)].

*Nonarchimedean case*. Take $\varepsilon > 0$. Note that any two (closed-open) $\varepsilon$-radius spheres $B(\alpha) = \{\mu \in L \mid |\mu - \alpha| < \varepsilon\}$ are equal or disjoint. Choose an indexed set $(\alpha_i)_{i \in I}$ in $f(T)$ such that $B(\alpha_i)_{i \in I}$ is disjoint and covers $f(T)$. We claim $I$ is finite. In fact, assume, to the contrary, that $I$ is infinite. Let $A_i = \bigcup_{j \neq i} f^{-1}(B(\alpha_j))$. Since the family of closed-open sets $(A_i)_{i \in I}$ has the finite intersection property, there exists $M \in \mathfrak{M}$ such that $A_i \in Z[M]$ for any $i \in I$. On the other hand, since $f \in C_0$ there exists $\lambda \in L$ such that $(f - \lambda) \in M$ and hence $Z(f - \lambda) \cap A_i \neq \varnothing$ for any $i \in I$, which is a contradiction. Thus $I$ is finite and so $f(T)$ is precompact.

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Now we will prove that $f(T)$ is compact. Endow $\mathfrak{M}$ with the Zariski topology (also called the Stone topology) and identify any $t \in T$ with the fixed maximal ideal $M_t = \{f \in C \mid f(t) = 0\}$. Then, by virtue of the Gelfand-Kolmogoroff theorem and its ultraregular analogue (see [**5**, 7.3] and [**1**]), $\mathfrak{M}$ can be considered as the Stone-Čech compactification of $T$ in the real case and as the Banaschewski one in the nonarchimedean case. Since $f(T)$ is a precompact set, $f$ can be uniquely extended to a continuous function $f^\beta$: $\mathfrak{M} \to L$. Since $f \in C_0$, for each $M \in \mathfrak{M}$ one has that $f^\beta(M) = \lambda$ if $(f - \lambda) \in M$; hence $(f - f^\beta(M)) \in M$ and $Z(f - f^\beta(M)) \neq \varnothing$, so for each $M \in \mathfrak{M}$ there exists $t \in T$ such that $f^\beta(M) = f(t)$. This shows that $f^\beta(\mathfrak{M}) \subset f(T)$, and obviously $f(T) \subset f^\beta(\mathfrak{M})$, thus $f(T) = f^\beta(\mathfrak{M})$ is a compact set.

PROPOSITION 2. $C_F \subset C_0$.

PROOF. Take $f \in C_F$ and let $K$ be a compact set such that $f(T - K) = \{\lambda_1, \ldots, \lambda_n\}$. If $M = M_t$ is a fixed maximal ideal of $C$ then $(f - f(t)) \in M$; if $M$ is a free maximal ideal of $C$ then $\Pi(f - \lambda_i) \in C_K \subset M$, so there exists $1 \leqslant i \leqslant n$ such that $(f - \lambda_i) \in M$. Hence $f \in C_0$ and so $C_F \subset C_0$.

THEOREM. *Assume that $T$ is paracompact and locally compact. Then for $f \in C$ the following statements are equivalent*: (1) $f \in C_F$, (2) *$f$ has finite range on every discrete closed subset of $T$, and* (3) $f \in C_0$.

PROOF. It is evident that (1) $\Rightarrow$ (2). From Proposition 2 it follows that (1) $\Rightarrow$ (3). Now let $f \in C$ and assume that $f(T - K)$ is infinite for every compact set $K \subset T$, which implies, in particular, that $T$ is not compact. We claim that there exists a discrete closed subset of $T$ in which $f$ has infinite range and that $f \notin C_0$. Note that the theorem follows from this claim. For the proof of the claim we will distinguish two cases.

*Real case*. First, we will consider the case in which $T$ is $\sigma$-compact (see Bourbaki [**3**, p. 68]). Take a sequence $(U_n)$ of relatively compact open sets such that $\overline{U}_n \subset U_{n+1}$ and $T = \bigcup U_n$. From the assumptions on $f$ there exists a sequence $(t_n)$, $t_n \in U_{i_n} - \overline{U}_{i_{n-1}}$ for some increasing sequence $(i_n)$, such that $f(t_n) \neq f(t_m)$ for $n \neq m$. For convenience, we set $V_n = U_{i_n}$. It is evident that the set $\{t_n \mid n \in \mathbf{N}\}$ is a discrete closed subset of $T$ on which $f$ has infinite range. To see that $f \notin C_0$ set $K_n = \overline{V}_{3n} - V_{3n-1}$, $L_n = \overline{V}_{3n-2}$. Since $K_n$ is compact, $L_n$ is closed, $K_n \cap L_n = \varnothing$ and $T$ is completely regular, there exists a continuous function $g_n$: $T \to [0, 1]$ such that $g_n|_{K_n} = 0$ and $g_n|_{L_n} = 1$. On the other hand, there exists another continuous function $l_n$: $\mathbf{R} \to [0, 1]$ such that $f(t_{3n}) = Z(l_n)$ and so $Z(f - f(t_{3n})) \cap Z(g_n) = Z(h_n)$ where $h_n = \sup(l_n \circ f, g_n)$. Now set $D_k = \bigcup_{k \leqslant n} Z(h_n)$ and $d_k = \inf_{k \leqslant n} h_n$. For any $m \in \mathbf{N}$ there exists $i(m) \in \mathbf{N}$ such that $h_n|_{V_m} = 1$ if $n \geqslant i(m)$, so $d_k|_{V_m} = \inf_{k \leqslant n \leqslant i(m)} h_n$. Hence $d_k \in C$ and $Z(d_k) = D_k$. Since the family of all $z$-sets $D_k$ has the finite intersection property, there exists $M \in \mathfrak{M}$ such that $D_k \in Z[M]$ for any $k \in \mathbf{N}$. If $f \in C_0$ then $(f - \lambda) \in M$ for some $\lambda \in \mathbf{R}$ and consequently $Z(f - \lambda) \cap D_k \neq \varnothing$ for all $k$, which is a contradiction. Thus $f \notin C_0$ and the claim is proved for $T$ $\sigma$-compact.

For general $T$ the proof is reduced to the above case if we show that $T$ contains a compact closed-open set $T'$ with the property that $f(T - K)$ is infinite for every compact $K \subset T'$. Since $T$ is a paracompact locally compact Hausdorff space, $T$ is the disjoint union of a family $(T_i)_{i \in I}$ of $\sigma$-compact open subsets of $T$. If some $T_i$ has the above stated property, set $T' = T_i$. Otherwise, take a sequence $(t_{i_n})$ such that $t_{i_n} \in T_{i_n}$ and $f(t_{i_n}) \neq f(t_{i_m})$ for $n \neq m$, and set $T' = \bigcup T_{i_n}$. This completes the proof of the real case.

*Nonarchimedean case*. From the topological assumptions on $T$ it follows that $T$ is the disjoint union of a family $(T_i)_{i \in I}$ of compact-open subsets of $T$. From the assumptions on $f$ there exists $t_n \in T_{i_n}$, $n = 1, 2, \ldots,$ such that $i_n \neq i_m$ and $f(t_n) \neq f(t_m)$ for $n \neq m$. Since the range of $f$ is infinite over the discrete closed subset $\{t_n \mid n \in \mathbf{N}\}$, the proof will be completed if we show $f \notin C_0$. Define

$$h_k(t) = \begin{cases} f(t) - f(t_n), & t \in T_{i_n} \text{ and } n \geqslant k, \\ 1, & \text{otherwise.} \end{cases}$$

Then $h_k$ is continuous, and, by letting $D_k = Z(h_k)$, we may proceed as in the real case.

The hypothesis of paracompactness is not superfluous as the following example shows:

EXAMPLE 1 (SEE [**5**, p. 123]). Let $w_1$ be the first uncountable ordinal and let $W^*$ be the set of all ordinals less than $w_1 + 1$ endowed with the interval topology. Let $T^* = W^* \times \mathbf{N}^*$, where $\mathbf{N}^*$ denotes the one-point compactification $\mathbf{N} \cup \{w\}$ of $\mathbf{N}$, and let $T = T^* - \{t\}$, where $t = (w_1, w)$. $T$ is a pseudocompact locally compact Hausdorff space. Since $T$ is pseudocompact we have $C_0(T, \mathbf{R}) = C(T, \mathbf{R})$. The continuous function $f$ defined by $(\alpha, n) \to 1/n$, $(\alpha, w) \to 0$ belongs to $C_0(T, \mathbf{R})$, but $f \notin C_F(T, \mathbf{R})$.

The above example shows that, in general, the equality $C_K = \bigcap \{M \in \mathfrak{M} \mid M$ is free$\}$ does not hold (see [**5**, p. 123]). However, as a consequence of our theorem one has the following

COROLLARY. *If the space $T$ is paracompact and Hausdorff locally compact then $C_K = \bigcap \{M \in \mathfrak{M} \mid M$ is free$\}$.*

PROOF. If $f$ lies in every free maximal ideal of $C$ then $f \in C_0$, and according to the theorem one has $f \in C_F$. Let $K$ be a compact subset of $T$ such that $f(T - K) = \{\lambda_1, \ldots, \lambda_n\}$. From [**5**, p. 58], it follows that $Z(f - \lambda_i)$ is compact for $\lambda_i \neq 0$, $1 \leqslant i \leqslant n$. Hence $f \in C_K$.

REMARK (SEE [**2**, p. 20]). The Theorem and the Corollary are also true if $L$ is replaced by the valued field of complex numbers $\mathbf{C}$. This can be easily deduced from the fact that the maximal ideals of $C(T, \mathbf{R})$ are in 1-1 correspondence with the maximal ideals of $C(T, \mathbf{C})$ under the mapping $M \to M + iM$. (The inverse of this map is the map sending the maximal ideal $m$ of $C(T, \mathbf{C})$ into $\mathrm{re}(m)$, where $\mathrm{re}(m)$ denotes the collection of real parts $\mathrm{re}(G)$ of functions $G$ in $m$.)

Finally, we give an example in which $T$ is not a normal space but, nevertheless, the conclusion of the Theorem is true.

EXAMPLE 2 (SEE [**5**, EXERCISE 8L]). Let $w_1$ and $W^*$ be as in Example 1. Let $T^* = W^* \times W^*$ and $T = T^* - \{(w_1, w_1)\}$. $T$ is a pseudocompact locally compact Hausdorff space, $T^*$ is the one-point compactification of $T$ and every function in $C(T, \mathbf{R})$ is constant on a deleted neighbourhood of $(w_1, w_1)$. Hence, $C(T, \mathbf{R}) = C_0(T, \mathbf{R}) = C_F(T, \mathbf{R})$, but $T$ is not normal.

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