

## INTO ISOMORPHISMS OF SPACES OF CONTINUOUS FUNCTIONS

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**ABSTRACT.** If  $X$  and  $Y$  are locally compact Hausdorff spaces and  $T$  is a linear map from an extremely regular subspace of  $C_0(X)$  into  $C_0(Y)$  such that  $\|T\|\|T^{-1}\| < 2$ , then  $X$  is a continuous image of a subset of  $Y$ .

**Introduction.** For a locally compact Hausdorff space  $X$ , we denote by  $C_0(X)$  the Banach space of all continuous complex valued functions defined on  $X$  which vanish at infinity, equipped with a usual sup norm. In case  $X$  is compact, we write  $C(X)$  instead of  $C_0(X)$ .

A well-known Banach-Stone theorem states that the existence of an isometry between the function spaces  $C_0(X)$  and  $C_0(Y)$  implies  $X$  and  $Y$  are homeomorphic.

D. Amir [1] and M. Cambern [3] independently generalized this theorem by proving that if  $C_0(X)$  and  $C_0(Y)$  are isomorphic under an isomorphism  $T$  satisfying  $\|T\|\|T^{-1}\| < 2$ , then  $X$  and  $Y$  must also be homeomorphic.

A generalization of another type of the Banach-Stone theorem was given by W. Holsztyński [5]. He proved that if there exists a linear isometry of  $C(X)$  into  $C(Y)$  then  $X$  is a continuous image of a closed subset of  $Y$ .

In [2] Y. Benyamini found, for compact metric spaces, a common generalization of the theorem of Amir-Camborn and Holsztyński. Namely he proved that if  $X$  is a compact metric space,  $0 < \epsilon < 1$ , and  $T$  is a linear homomorphism of  $C(X)$  into  $C(Y)$  satisfying  $\|f\| \leq \|Tf\| \leq (1 + \epsilon)\|f\|$  for all  $f \in C(X)$ , then first there is a continuous function  $\varphi$  from a closed subset  $Y_1$  of  $Y$  onto  $X$ ; second there is an isometry  $\Phi$  of  $C(X)$  into  $C(Y)$  such that

$$\Phi(f)(y) = f \circ \varphi(y) \quad \text{for } y \in Y_1,$$

and

$$\|\Phi - T\| \leq 3\epsilon.$$

An example given by Benyamini proves that the second part of his theorem in general does not hold for nonmetric spaces. The purpose of this note is to show that the first part of Benyamini's theorem is valid also for nonmetrizable spaces.

**The result.** According to [4], a closed linear subspace  $A$  of  $C_0(X)$  is said to be extremely regular if for each  $x_0$  in  $X$ , each neighbourhood  $U$  of  $x_0$  and each  $\epsilon > 0$ , there is a function  $f$  in  $A$  such that:

1.  $1 = f(x_0) = \|f\|$ ;
2.  $|f(x)| < \epsilon$  for all  $x \in X - U$ .

**THEOREM.** Let  $X, Y$  be locally compact spaces and let  $T$  be a linear isomorphism of an extremely regular subspace  $A$  of  $C_0(X)$  onto a closed linear subspace  $B$  of  $C_0(Y)$ .

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Received by the editors February 18, 1983.

1980 *Mathematics Subject Classification.* Primary 46E15; Secondary 46E25.

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If  $\|T\|\|T^{-1}\| = k < 2$  then there is a subset  $Y_1$  of  $Y$  and a continuous function  $\varphi$  from  $Y_1$  onto  $X$ . Moreover, if  $X$  is compact then  $\varphi$  can be extended to a continuous function  $\tilde{\varphi}$  from  $\bar{Y}_1$  onto  $X$ .

We divide the proof of the theorem into two lemmas, and we assume, without loss of generality, that  $\|T\| = k$  and  $\|T^{-1}\| = 1$ .

LEMMA 1. Assume  $A$  is an extremely regular subspace of  $C_0(X)$ . Then for each  $x_0$  in  $X$ , each neighbourhood  $U$  of  $x_0$  and each  $\epsilon > 0$  there is a function  $f$  in  $A$  such that:

1.  $1 = f(x_0) = \|f\|$ ;
2.  $|f(x)| < \epsilon$  for all  $x \in X - U$ ;
3.  $|f(x) - (\operatorname{Re} f(x))^+| < \epsilon$  for all  $x$  in  $X$ ,

where for a real number  $y$  we define

$$y^+ = \begin{cases} y & \text{if } y \geq 0, \\ 0 & \text{if } y \leq 0. \end{cases}$$

PROOF. Fix  $x_0$  in  $X$ ,  $\epsilon > 0$  and an open neighbourhood  $U$  of  $x_0$  and let  $n$  be a positive integer such that  $1/n < \epsilon/2$ .

We shall define by induction a sequence  $(f_k)_{k=1}^\infty \subset A$  and a descending sequence of neighbourhoods  $(U_k)_{k=1}^\infty$  of the point  $x_0$ .

To this end we set  $U_1 = U$  and let  $f_1$  be any function from  $A$  such that  $\|f_1\| = 1 = f_1(x_0)$  and  $|f_1(x)| < \epsilon/2$  for all  $x$  in  $X - U_1$ . Assume we have defined  $U_k$  and  $f_k$ ; then

$$U_{k+1} = \{x \in U_k : |f_k(x) - 1| < \epsilon/2\},$$

and as  $f_{k+1}$  we take any element of  $A$  such that  $\|f_{k+1}\| = 1 = f_{k+1}(x_0)$  and  $|f_{k+1}(x)| < \epsilon/2$  for all  $x$  in  $X - U_{k+1}$ . We define

$$f = \frac{1}{n} \sum_{k=1}^n f_k.$$

Points 1 and 2 of the lemma are evidently fulfilled. We shall check 3. If  $x \in X - U_1$ , then for all  $k \in \mathbb{N}$ ,  $|f_k(x)| < \epsilon/2$  and, consequently,  $|f(x)| < \epsilon/2$ , which gives 3. If  $x \in U_1$ , then denoting by  $k_0$  the greatest positive integer not greater than  $n$  such that  $x \in U_{k_0}$ , we find

$$\begin{aligned} \left| f(x) - \frac{k_0 - 1}{n} \right| &= \left| \frac{1}{n} \left( \sum_{j=1}^{k_0-1} (f_j(x) - 1) + f_{k_0}(x) + \sum_{j=k_0+1}^n f_j(x) \right) \right| \\ &\leq \frac{1}{n} \left[ (k_0 - 1) \frac{\epsilon}{2} + 1 + (n - k_0) \frac{\epsilon}{2} \right] = \frac{n-1}{n} \frac{\epsilon}{2} + \frac{1}{n} < \epsilon, \end{aligned}$$

which again gives 3.

For each  $x \in X$  ( $y \in Y$ ) let  $\delta_x$  ( $\delta_y$ ) denote the evaluation map on  $A$  (on  $C_0(Y)$ ); let  $\mu_x$  ( $\nu_y$ ) be any regular Borel measure on  $Y$  (on  $X$ ) such that

$$\mu_x(Tf) = f(x) \quad (\nu_y(f) = Tf(y)) \quad \text{for all } f \in A,$$

and

$$\operatorname{var}(\mu_x) = \|(T^{-1})^* \delta_x\| \quad (\operatorname{var}(\nu_y) = \|T^* \delta_y\|).$$

Fix any positive number  $M$  such that  $1/k > M > 1/2$ , and for  $x \in X$ , let  $\tilde{S}_x$  be the set of all points  $y \in Y$  such that  $|\nu_y(\{x\})| \geq M\|T^*\delta_y\|$ , and  $S_x$  be the set of all points  $y$  from  $\tilde{S}_x$  such that  $\|\delta_y|_{TA}\|$  (i.e. the norm of functional  $\delta_y$  restricted to  $TA$ ) is not less than  $k/2$ .

Notice that because  $M > 1/2$  we have  $\tilde{S}_{x_1} \cap \tilde{S}_{x_2} = \emptyset$  if  $x_1 \neq x_2$ .

LEMMA 2. For each  $x$  in  $X$  we have  $S_x \neq \emptyset$  and

- (i) for any compact subset  $K$  of  $X$  the set  $\bigcup_{x \in K} \tilde{S}_x$  is a closed subset of  $Y$ ;
- (ii) the function  $\varphi: \bigcup_{x \in X} S_x \rightarrow X: \varphi(y) = x$ , if  $y \in S_x$ , is continuous.

PROOF. Let us first consider some general facts. If  $A$  is a closed subspace of  $C_0(X)$ , then any functional  $F \in A^*$  can be extended, with the same norm, to a regular Borel measure  $\mu_F$  on  $X$ . On the other hand,  $A$  can be regarded as a subspace of the space of continuous functions on  $A^*$  with the weak  $*$  topology, and any such measure  $\mu_F$  can be regarded as a regular Borel measure defined on a closed subset of  $A^*$ . Notice that this is independent of the fact whether or not  $A$  separates points of the set  $X$ . Now let  $T$  be an isomorphism between two subspaces  $A$  and  $B$  of the spaces  $C_0(X)$  and  $C_0(Y)$ , respectively. Regarding  $A$  and  $B$  in the above way, for any functional  $F \in A^*$  we can first find a measure  $\mu_G$  on  $Y \subset B^*$  where  $G = (T^{-1})^*(F)$  and then a measure  $\xi_G$  on  $T^*(Y) \subset A^*$ :

$$\xi_G(K) = \mu_G((T^{-1})^*(K)) \text{ for any Borel subset } K \text{ of } A^*.$$

The measure  $\xi_G$  represents the same functional  $F$  on  $A$  and  $\text{var}(\xi_G) = \text{var}(\mu_G) = \|(T^{-1})^*(F)\|$ .

In the sequel we shall use the same notation for a function  $f \in A \subset C_0(X)$  and for the corresponding continuous function on  $A^*$  if no misunderstanding is likely to occur.

We now return to our situation and put  $F = \delta_{x_0}$ . We get a measure  $\xi_{x_0} = \xi_{(T^{-1})^*\delta_{x_0}}$  concentrated in  $T^*(Y) \subset A^*$  such that

$$(1) \quad f(x_0) = \delta_{x_0}(f) = (T^{-1})^*(\delta_{x_0})(Tf) = \mu_{x_0}(Tf) = \xi_{x_0}(f)$$

for all  $f$  in  $A$ , and

$$(2) \quad \text{var}(\xi_{x_0}) = \|(T^{-1})^*(\delta_{x_0})\| = \text{var}(\mu_{x_0}).$$

Fix  $\epsilon > 0$ . By the definition of  $\mu_{x_0}$  we have

$$\mu_{x_0}(\{y \in Y: \|\delta_y|_{TA}\| < 1\}) = 0,$$

so by regularity of  $\mu_{x_0}$ , there is a compact subset  $Y'$  of  $\{y \in Y: \|\delta_y|_{TA}\| > k/2\}$  such that  $|\mu_{x_0}(Y - Y')| < \epsilon$ . From (1) and (2) there exists, for any  $f$  in  $A$  such that  $\|f\| = 1 = f(x_0)$ , an  $F \in \text{supp } \xi_{x_0} \cap T^*(Y')$  with the property that  $|F(f)| \geq 1 - 2\epsilon$ .

Let a net  $(U_\alpha)_{\alpha \in \Gamma}$  constitute a neighbourhood basis at  $x_0$  indexed in the obvious way, and let  $(f_\alpha)$  be a net of functions from  $A$  obtained by Lemma 1, i.e. such that for all  $\alpha \in \Gamma$ :

1.  $\|f_\alpha(x_0)\| = 1 = f_\alpha(x_0)$ ;
2.  $|f_\alpha(x)| < \epsilon$  for all  $x \in X - U_\alpha$ ;
3.  $|f_\alpha(x) - (\text{Re } f_\alpha(x))^+| < \epsilon$  for all  $x \in X$ .

Let  $F_\alpha = T^*(\delta_{y_\alpha})$  be any functional such that  $y_\alpha \in Y'$  and

$$(3) \quad |F_\alpha(f_\alpha)| \geq 1 - 2\epsilon.$$

Let us decompose the measure  $\nu_{y_\alpha}$  as follows:  $\nu_{y_\alpha} = \nu_\alpha^1 + \nu_\alpha^2$ , where  $\nu_\alpha^1 = \nu_{y_\alpha|U_\alpha}$ ,  $\nu_\alpha^2 = \nu_{y_\alpha|X-U_\alpha}$ . We can assume the nets  $(\nu_\alpha^1)$  and  $(\nu_\alpha^2)$  are weak  $*$  convergent to measures  $\nu^1$  and  $\nu^2$ , respectively, and  $(y_\alpha)$  converges to  $y_0 \in Y'$ .

From (3) and the definition of  $\nu_{y_\alpha}$  and  $f_\alpha$  we have

$$(4) \quad |\nu_{y_\alpha}(U_\alpha)| \geq 1 - 6\epsilon.$$

Hence  $\nu^1 = \lambda_1 \delta_{x_0}$  where  $|\lambda_1| \geq 1 - 6\epsilon$ . On the other hand the measure  $\nu^2$  can be represented in the form

$$\nu^2 = \lambda_2 \delta_{x_0} + \Delta \nu^2 \quad \text{where } \Delta \nu^2(\{x_0\}) = 0.$$

Hence

$$\nu^0 = \nu^1 + \nu^2 = (\lambda_1 + \lambda_2) \delta_{x_0} + \Delta \nu^2.$$

From (4),

$$|\nu_\alpha(U_\alpha)| \geq 1 - 6\epsilon \geq \frac{1 - 6\epsilon}{k} (|\nu_\alpha|(U_\alpha) + \text{var}(\nu_\alpha^2)).$$

Hence, if  $1 - 6\epsilon \geq M \cdot k$  then

$$(5) \quad |\lambda_1| \geq M(|\lambda_1| + \text{var}(\nu^2)) = M(|\lambda_1| + |\lambda_2| + \text{var}(\Delta \nu^2)).$$

Notice now that, since  $1/2 < M < 1$ , for any  $\lambda_1, \lambda_2 \in \mathbb{C}$  we have

$$(6) \quad |\lambda_1 + \lambda_2| - |\lambda_1| \geq M(|\lambda_1 + \lambda_2| - |\lambda_1| - |\lambda_2|).$$

Adding (5) and (6) we get

$$(7) \quad |\lambda_1 + \lambda_2| \geq M(|\lambda_1 + \lambda_2| + \text{var}(\Delta \nu^2)) \geq M \text{var}(\nu^0) \geq M \|T^* \delta_{y_0}\|.$$

Now let  $\nu$  be any regular measure on  $X$  which represents  $T^* \delta_{y_0}$ . Since  $A$  is an extremely regular subspace of  $C_0(X)$ , we have  $|\nu(\{x_0\})| \geq |\lambda_1 + \lambda_2|$ ; hence by (7),

$$|\nu(\{x_0\})| \geq M \|T^* \delta_{y_0}\|$$

and this proves  $y_0 \in S_{x_0}$ .

(i) Now fix a compact subset  $K$  of  $X$  and let  $y_\alpha \in S_{x_\alpha}$ ,  $x_\alpha \in K$  and  $y_\alpha \rightarrow y_0 \in Y$ . Recall that  $\nu_{y_\alpha}$  denotes a regular Borel measure on  $Y$  which represents the functional  $T^* \delta_{y_\alpha}$  and is such that  $\text{var}(\nu_{y_\alpha}) = \|T^* \delta_{y_\alpha}\|$ .

By our assumption

$$(8) \quad \nu_{y_\alpha} = \lambda_\alpha \delta_{x_\alpha} + \Delta \nu_\alpha \quad \text{where } |\lambda_\alpha| \geq M(|\lambda_\alpha| + \text{var}(\Delta \nu_\alpha)).$$

Without loss of generality we can assume the nets  $(\lambda_\alpha)$ ,  $(x_\alpha)$  and  $(\Delta \nu_\alpha)$  are convergent, in appropriate topologies, to  $\lambda_0 \in \mathbb{C}$ ,  $x_0 \in K$  and to a measure  $\Delta \nu_0 = \lambda' \delta_{x_0} + \rho$ , where  $\rho(\{x_0\}) = 0$ , respectively.

From (8), we have

$$|\lambda_0| \geq M(|\lambda_0| + |\lambda'| + \text{var}(\rho));$$

hence, as in the proof of previous part we get

$$|\lambda_0 + \lambda'| \geq M(|\lambda_0 + \lambda'| + \text{var}(\rho)).$$

Next we get  $|\nu_{y_0}(\{x_0\})| \geq M \text{var}(\nu_{y_0})$  and, hence,  $y_0 \in \bigcup_{x \in K} \tilde{S}_x$ .

(ii) Let  $y_\alpha \in S_{x_\alpha}$  for  $\alpha \in \Gamma$  and  $y_\alpha \rightarrow y_0 \in S_{x_0}$ . To prove the continuity of  $\varphi$  it is sufficient to show that  $x_\alpha$  is neither convergent to any  $x_1 \neq x_0$  nor divergent to infinity (in the case  $X$  is noncompact).

Assume first that  $x_\alpha \rightarrow x_1 \neq x_0$ . We have

$$\nu_{y_\alpha} = \lambda_\alpha \delta_{x_\alpha} + \Delta\nu_\alpha \quad \text{for } \alpha \in \Gamma \cup \{0\} \cup \{1\},$$

where  $|\lambda_\alpha| \geq M(|\lambda_\alpha| + \text{var}(\Delta\nu_\alpha))$ . Assume without loss of generality that  $\lambda_\alpha \rightarrow \lambda'$ ,  $\Delta\nu_\alpha \rightarrow \Delta\nu'$ . Since  $T^*\delta_{y_\alpha}$  tends to  $T^*\delta_{y_0}$  in weak \* topology, it follows that the measures  $\lambda'_1 \delta_{x_1} + \Delta\nu'$  and  $\lambda_0 \delta_{x_0} + \Delta\nu_0$  represent the same functionals on  $A$ , but this is impossible since  $|\lambda'_1| + |\lambda_0| > \text{var}(\Delta\nu_1) + \text{var}(\Delta\nu_0)$ , and  $A$  is extremely regular.

Now assume  $X$  is noncompact and  $x_\alpha$  is divergent to infinity, and use the same notation as above. The net  $\lambda_\alpha \delta_{x_\alpha}$  tends to zero so the measure  $\Delta\nu'$  represents the functional  $T^*\delta_{y_0}$ , but

$$\text{var}(\Delta\nu') \leq \liminf \text{var}(\Delta\nu_\alpha) < k/2,$$

and this contradicts our assumption

$$S_{x_0} \subset \{y \in Y : \|\delta_y|_{TA}\| > k/2\}.$$

To end the proof of the theorem put

$$\tilde{\varphi}: \bigcup_{x \in X} \tilde{S}_x \rightarrow X: \tilde{\varphi}(y) = x \quad \text{if } y \in \tilde{S}_x.$$

For a compact space  $X$  the inclusion  $\bar{Y}_1 \subset \bigcup_{x \in X} \tilde{S}_x$  and the continuity of  $\tilde{\varphi}$  are immediate consequences of Lemma 2(i).

REMARK. The assertion of the Theorem cannot be strengthened to the effect “ $Y_1$  is a closed subset of  $Y$ ”. A simple counterexample is obtained by taking  $A$  to be the space  $c_0$  of all infinite sequences tending to zero,  $C_0(Y)$  the space  $c$  of all convergent sequences and  $T$  the natural isometric embedding of  $c_0$  into  $c$ .

## REFERENCES

1. D. Amir, *On isomorphism of continuous function spaces*, Israel J. Math. **3** (1965), 205–210.
2. Y. Benyamini, *Small into-isomorphisms between spaces of continuous functions*, Proc. Amer. Math. Soc. **83** (1981), 479–485.
3. M. Cambern, *On isomorphisms with small bound*, Proc. Amer. Math. Soc. **18** (1967), 1062–1066.
4. B. Cengiz, *A generalization of the Banach-Stone theorem*, Proc. Amer. Math. Soc. **40** (1973), 426–430.
5. W. Holsztyński, *Continuous mappings induced by isometries of spaces of continuous functions*, Studia Math. **26** (1966), 133–136.

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