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ON DIVERGENT LACUNARY TRIGONOMETRIC SERIES

L. THOMAS RAMSEY

ABSTRACT. Let $S = \{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\lambda_{n+1}/\lambda_n \geqslant q > 1$ for all n. If $q > 8$ then any divergent series with frequencies in S has its real part diverging (uniformly) to $+\infty$ on a set of positive logarithmic capacity. It is necessary that $q > 2$. A new sufficient condition for the generalized capacity of a set to be positive is developed and then applied in the proof.

In [3] Csordas, Lohwater and Ramsey prove the following theorem. We use $\exp(t)$ to denote e^{it}.

THEOREM 1. *Let $S = \{\lambda_n\}_{n=1}^{\infty}$ be a set of positive real numbers such that $\lambda_{n+1}/\lambda_n \geqslant q > 1$ for all n. Let $\sum_{n=1}^{\infty} |c_n| = +\infty$ with $\sup_n |c_n| < \infty$. Then*

$$\lim_{N\to\infty}\left(\operatorname{Re}\sum_{n=1}^{N} c_n \exp(\lambda_n x)\right) = +\infty$$

for all x in some set of positive logarithmic capacity.

In this present paper we address the question of whether $\sup_n |c_n| < \infty$ is a necessary hypothesis.

In part the answer is "yes". The series $\sum_{n=1}^{\infty} 4^n \cos(2^n x)$ diverges to $+\infty$ on exactly the countable set of x such that for some positive integer n, $2^n x = 0 \bmod 2\pi$. However, for cosine series the theorem remains true with $q > 4$ and for general series with $q > 8$. The proof depends upon a new sufficient condition for a set to have positive generalized capacity.

Let $\varphi\colon R^+ \to R$ be a continuous decreasing function with $\lim_{t\to 0^+} \varphi(t) = +\infty$. Let φ be integrable on $[0, 1]$, and set $\Phi(t) = \int_0^t \varphi(s)\,ds$. We define the φ-capacity of a Borel set $E \subseteq R$ as in [**1** and **2**]. Let μ be a probability measure on E and consider

$$u(x) = \int_R \varphi(|x-y|)\,d\mu(y) \quad \text{for } y \in R.$$

E is said to have positive φ-capacity if $\sup_{x\in R} u(x) < \infty$ for at least one measure μ.

The theorem below concerns Cantor sets C constructed as follows. $C = \bigcap_{n=1}^{\infty}(\bigcup I_n)$ where each I_n is a set of closed, disjoint intervals of $[0, 1]$ of equal length d_n. Each interval I in I_n contains exactly $K_{n+1} \geqslant 2$ intervals from I_{n+1}. For ease of notation, assume also that the number of intervals in I_1 is $K_1 \geqslant 2$. By the Lebesgue measure on C we mean that probability measure μ satisfying $\mu(I) = A_n^{-1}$

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for I in I_n, where $A_n = \prod_{p=1}^{n} K_p$. Given any interval J of length d_n, clearly no more than two intervals from I_n can intersect it. Thus $\mu(J) \leq 2A_n^{-1}$, an estimate which will be used later. This construction of the Cantor set C is more general than that of [**1**] in two respects. In [**1**] the intervals of I_{n+1} inside a single interval of I_n are equally spaced. An examination of the proof of [**1**] reveals that this is unnecessary. More importantly, in [**1**] K_{n+1} is a constant, independent of n.

THEOREM 2. *If* $\sum_{n=1}^{\infty}[\Phi(d_n)/(d_{n+1} \cdot A_{n+1})] < \infty$, *then* C *has positive* φ*-capacity. In fact,* $\sup_{x\in R} u(x) < \infty$ *for* $u(x) = \int \varphi(|x-y|)\,d\mu(y)$ *with the Lebesgue measure* μ *on* C.

PROOF. Let μ be the Lebesgue measure on C and let $x \in R$. Set

$$J_p = [x - d_p, x + d_p] \setminus [x - d_{p+1}, x + d_{p+1}] \quad \text{for } p \geq 1.$$

Make $J_0 = R \setminus [x - d_1, x + d_1]$. Then

$$\begin{aligned} u(x) &= \int_R \varphi(|x-y|)\,d\mu(y) = \sum_{p=0}^{\infty} \int_{J_p} \varphi(|x-y|)\,d\mu(y) \\ &\leq \varphi(d_1)\mu(J_0) + \sum_{p=1}^{\infty} \int_{J_p} \varphi(|x-y|)\,d\mu(y). \end{aligned}$$

For each $p \geq 1$ we estimate $\int_{J_p} \varphi(|x-y|)\,d\mu(y)$. Partition $[x + d_{p+1}, x + d_p]$ with $x_0 = x + d_{p+1}$, $x_1 = x + 2d_{p+1}, \ldots, x_{n_p} = x + n_p d_{p+1}$ where $x + n_p d_{p+1} < x + d_p \leq x + (n_p + 1)d_{p+1}$. Because φ is decreasing and nonnegative near zero, for p large

$$\begin{aligned} \int_{x+d_{p+1}}^{x+d_p} \varphi(|x-y|)\,d\mu(y) &\leq \int_{x+d_{p+1}}^{x+(n_p+1)d_{p+1}} \varphi(|x-y|)\,d\mu(y) \\ &\leq \sum_{j=1}^{n_p} \varphi(jd_{p+1})2A_{p+1}^{-1} = \left(\frac{2}{d_{p+1}A_{p+1}}\right) \sum_{j=1}^{n_p} \varphi(jd_{p+1})d_{p+1}. \end{aligned}$$

Note that $\sum_{j=1}^{n_p} \varphi(jd_{p+1})d_{p+1} \leq \int_0^{n_p d_{p+1}} \varphi(|y|)\,dy$, because φ is decreasing. This last is $\Phi(n_p d_{p+1})$. Note that $n_p d_{p+1} \leq d_p$. Also, if p is large enough, d_p is small and φ is positive on $[0, d_p]$. Then $\Phi(n_p d_{p+1}) \leq \Phi(d_p)$ and we have

$$\int_{x+d_{p+1}}^{x+d_p} \varphi(|x-y|)\,d\mu(y) \leq \frac{2\Phi(d_p)}{d_{p+1}A_{p+1}}.$$

By treating $\int_{x-d_p}^{x-d_{p+1}} \varphi(|x-y|)\,d\mu(y)$ in a similar manner, we obtain

$$\int_{J_p} \varphi(|x-y|)\,d\mu(y) \leq \frac{4\Phi(d_p)}{d_{p+1}A_{p+1}}.$$

This completes our estimate. Finally

$$u(x) \leq \varphi(d_1)\mu(J_0) + \sum_{p=1}^{\infty} 4\Phi(d_p)/(d_{p+1}A_{p+1}) < \infty,$$

with a bound independent of x.

COROLLARY 3. *Let $\varphi(t) = -\log t$. Let C be constructed as above; let A_n, d_n, and K_n be defined as above. If for some $\alpha > 0$, $d_{n+1}A_{n+1} \geqslant \alpha d_n A_n$, the logarithmic capacity of C is positive.*

PROOF OF COROLLARY. We apply the theorem. Here $\varphi(t) = -t\log t + t$. The test series is

$$\sum_{n=1}^{\infty} (-d_n \log d_n + d_n)/(d_{n+1}A_{n+1}). \tag{1}$$

Because $d_{n+1}A_{n+1} \geqslant \alpha d_n A_n$, this series is dominated by

$$\alpha^{-1} \sum_{n=1}^{\infty} (1 - \log d_n)/(A_n). \tag{2}$$

By induction we have $d_n \geqslant \alpha^{n-1}(d_1 A_1)/(A_n)$, so that for some constant C we have

$$1 - \log d_n \leqslant (n-1)(-\log \alpha) - C + \log A_n. \tag{3}$$

Note that $A_n \geqslant 2^n$. Because $(\log x)/x$ is decreasing for $x \geqslant e$, for $n \geqslant 2$ we have $\log(A_n)/A_n \leqslant \log(2^n)/2^n$. Thus the series (2) is dominated by the convergent series (4):

$$\alpha^{-1}\left\{(1 - \log d_1)/A_1 + \sum_{n=2}^{\infty} [((n-1)(-\log\alpha) - C + n\log 2)/2^n]\right\}. \tag{4}$$

That completes the proof.

We next apply Corollary 3 to lacunary divergent trigonometric series in the proof of Theorem 4.

THEOREM 4. *Let ρ satisfy $0 < \rho < 1$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be a lacunary sequence of positive integers such that $\lambda_{k=1}/\lambda_k \geqslant 8\rho^{-1}$ for all k. Let $\sum_{k=1}^{\infty} |c_k| = +\infty$. Then there is some set $C \subset [0, 2\pi]$ of positive logarithmic capacity on which*

$$\operatorname{Re}\left\{\sum_{k=1}^{N} c_k \exp(\lambda_k x)\right\} \geqslant K_\rho\left(\sum_{k=1}^{N} |c_k|\right) \quad \textit{for all } N \tag{5}$$

where $K_\rho = \cos(\rho(\pi/2))$.

PROOF. C shall be a Cantor set as constructed above, $C = \bigcap_n(\bigcup I_n)$, to which we shall apply Corollary 3. We shall have

$$\operatorname{Re}\{c_n \exp(\lambda_n x)\} \geqslant K_\rho(|c_n|) \quad \text{for } x \text{ in } I_n. \tag{6}$$

Each interval in I_n shall have length $\rho(\pi/2)\lambda_n$. The sets I_n shall be chosen inductively.

Write $c_1 = \exp(\varphi_1)|c_1|$, φ_1 real. Note that $\operatorname{Re}\{c_1\exp(\lambda_1 x)\} \geqslant K_\rho|c_1|$ if and only if $|x - ((k\cdot 2\pi - \varphi_1)/\lambda_1)| \leqslant \rho(\pi/2)\lambda_1$ for some integer k. Let I_1 consist of one interval of length $d_1 = \rho(\pi/2)/\lambda_1$ inside $[0, 2\pi]$ such that $\operatorname{Re}\{c_1\exp(\lambda_1 x)\} \geqslant K_\rho|c_1|$ on that interval.

For the inductive hypothesis assume that we have chosen $\{I_k\}_{k=1}^{n}$, a sequence of nested sets of intervals as described above, with the intervals of I_k having equal length $d_k = \rho(\pi/2)\lambda_k$ and $d_{k+1} \cdot A_{k+1} \geqslant (\rho/8)d_k \cdot A_k$. We then choose I_{n+1} as follows. Write c_{n+1} as $\exp(\varphi_{n+1})|c_{n+1}|$. Note that $\operatorname{Re}\{c_{n+1}\exp(\lambda_{n+1}x)\} \geqslant K_\rho |c_{n+1}|$ on intervals of the form

$$[(p \cdot 2\pi - \varphi_{n+1})/\lambda_{n+1} - \rho(\pi/2)/\lambda_{n+1}, (p \cdot 2\pi - \varphi_{n+1})/\lambda_{n+1} + \rho(\pi/2)/\lambda_{n+1}],$$

for arbitrary integers p. Let K_{n+1} be the greatest integer in

$$\rho(\pi/2)/\lambda_n/(2\pi/\lambda_{n+1}) = (\rho/4)(\lambda_{n+1}/\lambda_n) \geqslant 2.$$

Thus each interval of I_n (because its length is $\rho(\pi/2)\lambda_n$) contains at least $K_{n+1} - 1$ consecutive intervals of the form

$$[(p \cdot 2\pi - \varphi_{n+1})/\lambda_{n+1}, ((p+1)2\pi - \varphi_{n+1})/\lambda_{n+1}] \tag{7}$$

where p is an integer. Inside each interval of I_n we take (for I_{n+1}) the initial segments of length $\rho(\pi/2)/\lambda_{n+1}$ of those intervals of form (7). We also include in I_{n+1} the last segment of length $\rho(\pi/2)/\lambda_{n+1}$ of the last interval of form (7) inside of each interval of I_n. That completes our choice of I_{n+1}. There remains to be checked whether $d_{n+1} \cdot A_{n+1} \geqslant (\rho/8)d_n \cdot A_n$. We have

$$d_{n+1} \cdot A_{n+1} = (\rho(\pi/2)/\lambda_{n+1}) \cdot K_{n+1} \cdot A_n.$$

Recall that K_{n+1} is the greatest integer in $\rho(\pi/2)/\lambda_n/(2\pi/\lambda_{n+1}) = (\rho/4)(\lambda_{n+1}/\lambda_n)$. It is easy to check that the greatest integer in $(\rho/4)y$ is no less than $(\rho/8)y$ for $y \geqslant 8/\rho$. It follows easily that $d_{n+1} \cdot A_{n+1} \geqslant (\rho/8)d_n \cdot A_n$. That completes the proof of this theorem.

REMARK 5. In the case of cosine series the previous argument can be simplified because one may assume φ_k is 0 for all k. In that case it suffices to assume that $\lambda_{k+1}/\lambda_k \geqslant 4/\rho$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII AT MANOA, HONOLULU, HAWAII 96822