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THE JUMP INVERSION THEOREM FOR Q_{2n+1}-DEGREES

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ABSTRACT. Assuming projective determinacy we extend Friedberg's Jump Inversion theorem to Q_{2n+1}-degrees, after noticing that it fails for Δ^1_{2n+1}-degrees.

0. Preliminaries. We list some results from the theory of countable analytical sets and Q-theory. For a more extensive account, including proofs, see [**2** and **5**]. Some familiarity with forcing in the analytical hierarchy is assumed; consult [**3** and **4**].

DEFINITION 0.1 (PD). C_{2n+1} is the largest countable Π^1_{2n+1} set of reals.

DEFINITION 0.2 (PD). C_{2n+2} is the largest countable Σ^1_{2n+2} set of reals.

We mention some of their properties: C_{2n+2} is the set of reals that are recursive in some element of C_{2n+1}. The set C_m is made up of Δ^1_m-degrees (a Δ^1_m-degree is a set of reals that is an equivalence class for the equivalence relation $\alpha \equiv_{\Delta_m} \beta \Leftrightarrow \alpha \in \Delta^1_m(\beta)$ and $\beta \in \Delta^1_m(\alpha)$). The Δ^1_m-degrees in the set C_m are well-ordered by $\alpha \leqslant_{\Delta_m} \beta \Leftrightarrow \alpha \in \Delta^1_m(\beta)$.

DEFINITION 0.3. Given $S \subset \omega^\omega$ let $H_{2n+1}(S) = \{\alpha\colon \forall\beta \in S\ (\alpha \in \Delta^1_{2n+1}(\beta))\}$; we call it the *hull* of S. If S is a nonempty Σ^1_{2n+1} set then $H_{2n+1}(S)$ is called a Σ^1_{2n+1}-hull. We let know Q_{2n+1} = the union of all Σ^1_{2n+1}-hulls.

We have, assuming PD: The set Q_{2n+1} is Π^1_{2n+1}. Every Σ^1_{2n+1}-hull is Π^1_{2n+1}-bounded (this means that if $R(\alpha, x)$ is Π^1_{2n+1} this so is $\exists\alpha \in H_{2n+1}(S)R(\alpha, x)$). The set Q_{2n+1} is the largest Σ^1_{2n+1}-hull, and the largest Π^1_{2n+1}-bounded set. Relativizing to an arbitrary real β we may define the set $Q_{2n+1}(\beta)$. We define also $\alpha \leqslant_{Q_{2n+1}} \beta \Leftrightarrow \alpha \in Q_{2n+1}(\beta)$, and $\alpha \equiv_{Q_{2n+1}} \beta \Leftrightarrow \alpha \in Q_{2n+1}(\beta)$ and $\beta \in Q_{2n+1}(\alpha)$. This is an equivalence relation, and the equivalence classes are called Q_{2n+1}-degrees. The set C_{2n+1} consists of such degrees. The set Q_{2n+1} is the largest initial segment of C_{2n+1} closed under $\leqslant_{\Delta_{2n+1}}$; it consists of the Δ^1_{2n+1}-degrees in C_{2n+1} up to and not including the degree of the first nontrivial (i.e. non-Δ^1_{2n+1}) Π^1_{2n+1} singleton y_0^{2n+1}. Relativizing to α we have y_α^{2n+1}. If $\alpha \leqslant_{Q_{2n+1}} \beta$ then $y_\alpha^{2n+1} \leqslant_{\Delta_{2n+1}} y_\beta^{2n+1}$, and y_α^{2n+1} plays the role of the *jump* for Q_{2n+1}-degrees. The set Q_{2n+1} is closed under the Δ^1_{2n+1}-jump.

To obtain an *ordinal assignment* for the Q_{2n+1}-degrees we proceed as follows.

DEFINITION 0.4.

$$\begin{aligned}\lambda_{2n+1} &= \sup\{\xi\colon \xi \text{ is the length of a } \Sigma^1_{2n+1} \text{ wellfounded relation on } \omega^\omega\}\\ &= \sup\{\xi\colon \xi \text{ is the length of a } \Delta^1_{2n+1} \text{ prewellordering of } \omega^\omega\}.\end{aligned}$$

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Relativizing to α we obtain $\lambda_{2n+1}(\alpha)$. Finally

$$k_{2n+1}(\alpha) = \sup\{\lambda_{2n+1}(\langle \alpha, \beta \rangle) : \lambda_{2n+1}(\langle \alpha, \beta \rangle) < \lambda_{2n+1}(y_\alpha^{2n+1})\}.$$

Of course, λ_{2n+1} is the ordinal assignment for the Δ^1_{2n+1}-degrees, e.g. the Spector Criterion holds: $\mathbf{d} \leqslant_{\Delta_{2n+1}} \mathbf{e} \Rightarrow [\mathbf{d}' \leqslant_{\Delta_{2n+1}} \mathbf{e} \Leftrightarrow \lambda_{2n+1}(d) < \lambda_{2n+1}(e)]$. Now we have $\lambda_{2n+1}(\alpha) < k_{2n+1}(\alpha) < \lambda_{2n+1}(y_\alpha^{2n+1})$, $k_{2n+1}(\alpha)$ is invariant under $\equiv_{Q_{2n+1}}$, $\alpha \leqslant_{Q_{2n+1}} \beta \Rightarrow k_{2n+1}(\alpha) \leqslant k_{2n+1}(\beta)$, and the Spector Criterion is true for Q_{2n+1}-degrees: $\mathbf{d} \leqslant_{Q_{2n+1}} \mathbf{e} \Rightarrow [\mathbf{d}' \leqslant_{Q_{2n+1}} \mathbf{e} \Leftrightarrow k_{2n+1}(d) \leqslant k_{2n+1}(e)]$. Naturally $\mathbf{d}'$ is the degree of y_d^{2n+1}.

The relation $k_{2n+1}(\alpha) \leqslant k_{2n+1}(\beta)$ is Σ^1_{2n+1}.

1. Background and definitions. One of the early results in the theory of Turing degrees was the following:

FRIEDBERG JUMP INVERSION THEOREM [1]. *If* $\mathbf{b} \geqslant \mathbf{0}'$ *then there exists an* $\mathbf{a}$ *such that* $\mathbf{a}' = \mathbf{a} \vee \mathbf{0}' = \mathbf{b}$.

Of course $\mathbf{0}$ denotes the degree of the recursive sets, and $'$ denotes the Turing jump operation.

Next, the question was considered in the context of hyperdegrees. Let $\mathbf{0}$ denote the hyperdegree of the hyperarithmetical sets and $'$ the hyperjump. Does the above theorem hold? The answer is yes [6]:

JUMP INVERSION THEOREM FOR Δ^1_1-DEGREES. *If* $\mathbf{b} \geqslant \mathbf{0}'$ *then there exists an* $\mathbf{a}$ *such that* $\mathbf{a}' = \mathbf{a} \vee \mathbf{0}' = \mathbf{b}$.

A natural question now is: does the inversion theorem hold for Δ^1_{2n+1}-degrees? (We are assuming PD, needless to say). By a well-known argument Determinacy implies that there exists *some* cone on which inversion holds (a *cone*, by definition, is $\{\mathbf{a}\colon \mathbf{a} \geqslant \mathbf{b}\}$, and $\mathbf{b}$ is called the base of the cone). But what is the base of the cone? Is it again $\mathbf{0}'$? (I.e. the Δ^1_{2n+1}-jump of the degree of Δ^1_{2n+1} sets.) Surprisingly, the answer is no.

THEOREM (KECHRIS, UNPUBLISHED) (PD). *If* $n \geqslant 1$, *then no real in* C_{2n+2} *can be a base for a cone of inversion of the* Δ^1_{2n+1}-*jump*. ("*Cone of inversion*" *of course means that every member of the cone is the* Δ^1_{2n+1}-*jump of some* Δ^1_{2n+1}-*degree*.)

PROOF. For notational simplicity we let $2n + 1 = 3$. If a member of C_4 were abase then it would be recursive in a member of C_3, so without loss of generality assume a base b is in C_3. Consider the set $C = \{\alpha\colon \exists\beta \in Q_3(\alpha)\ (\beta \in C_3 \text{ and } \alpha \leqslant_{\Delta_3} \beta)\}$. It is a subset of C_4, and it is Π^1_3, because the quantification is bounded. So it is countable, and hence a subset of C_3. Since $b \in C_3$ everything $\geqslant b$ in C_3 is the Δ^1_3-jump of a member of C, thus a member of C_3. However the Δ^1_3-degrees in C_3 are wellordered with successor steps taken by the Δ^1_3-jump, so that a limit stage of this wellordering gives immediately a contradiction. (C_3 is closed under $\equiv_{Q_3}$, hence $\alpha' \in C_3 \Rightarrow \alpha \in C_3$, hence no limit level of C_3 is a Δ^1_3-jump.)

So the inversion theorem is a property of hyperdegrees that fails to generalize to Δ^1_{2n+1}-degrees, $n \geqslant 1$. Usually in such cases the validity of the property is restored if instead of Δ^1_{2n+1}-degrees we work with Q_{2n+1}-degrees. Indeed, it is the case that the jump inversion theorem holds for Q_{2n+1}-degrees, i.e. the base is again $\mathbf{0}'$. Moreover we can establish that the Q_{2n+1}-jump is never one-to-one.

JUMP INVERSION THEOREM FOR Q_{2n+1}-DEGREES (PD). *If* $\mathbf{c}$ *is a* Q_{2n+1}*-degree* $\geqslant \mathbf{0}'$ *then there exist* Q_{2n+1}*-degrees* $\mathbf{a}, \mathbf{b}$ *such that* $\mathbf{a} \vee \mathbf{b} = \mathbf{a}' = \mathbf{b}' = \mathbf{c}$.

The rest of the paper is devoted to the proof of this theorem.

2. The proof. For notational simplicity we work with $2n + 1 = 3$. First we establish a lemma.

LEMMA 2.1. *If* $\mathbf{0}' \not\leqslant \mathbf{b}$ (*i.e.* $k_3^0 = k_3^b$) *then* $\mathbf{b}' = \mathbf{b} \vee \mathbf{0}'$.

PROOF. By the Specter Criterion $\mathbf{0}' \not\leqslant \mathbf{b}$ iff $k_3^0 = k_3^b$. Now $k_3^0 < k_3^{\mathbf{b} \vee \mathbf{0}'}$, so again by the Spector Criterion $\mathbf{b}' \leqslant \mathbf{b} \vee \mathbf{0}'$. The opposite inequality is obvious.

PROOF OF THE THEOREM. The set $\{\alpha: k_3^\alpha = k_3^0$ and $\alpha \notin Q_3\}$ is Σ_3^1 and comeager. In fact there is a sequence $D_0, D_1, \ldots$ of dense open sets, $\{D_i\} \in \Delta_3^1(y_0)$, such that $\bigcap D_i \subset \{\alpha: k_3^\alpha = k_3^0$ and $\alpha \notin Q_3\}$. This is implicit in [3]; briefly, comeagerness is characterized by the Banach-Mazur game. Use the Game Formula to unfold it and make it Π_2^1; then the set of winning strategies is also Π_2^1, so there is a winning strategy recursive in y_0, by the Martin-Solovay basis theorem [5]. This gives the dense open sets.

We describe an inductive construction of reals a and b. Set $a_{-1} = b_{-1} = \varnothing$.

Inductive step. Suppose a_n, b_n have been constructed (they are finite sequences of integers). Consider the dense, open set D_{n+1} and extend a_n by a finite segment s, least in some fixed enumeration, so that the basic neighborhood defined by $\hat{a}_n s$ is contained in D_{n+1}. Extend $\hat{b}_n s$ by a finite segment t, least again, so that the basic neighborhood defined by $b_n\hat{}s\hat{}t$ is contained in D_{n+1}. Set now $a_{n+1} = a_n\hat{}s\hat{}t\hat{}\{c(n)\}$, $b_{n+1} = b_n\hat{}s\hat{}t\hat{}\{c(n) + 1\}$.

This completes the inductive step.

Let now $a = \bigcup a_n$, $b = \bigcup b_n$. Since $a, b \in \bigcap D_i$ we have by Lemma 2.1 that $\mathbf{a}' = \mathbf{a} \vee \mathbf{0}'$, $\mathbf{b}' = \mathbf{b} \vee \mathbf{0}'$. Now $\mathbf{a} \vee \mathbf{0}' \geqslant \mathbf{c}$, because using y_0 we may trace the construction of a and find all $c(n)$'s. Likewise $\mathbf{b} \vee \mathbf{0}' \geqslant \mathbf{c}$. However $\mathbf{a} \vee \mathbf{0}' \leqslant \mathbf{c}$, too, because $\mathbf{0}' \leqslant \mathbf{c}$ and the construction of a only needs y_0 and c. The same holds for b, and therefore we have $\mathbf{a}' = \mathbf{b}' = \mathbf{a} \vee \mathbf{0}' = \mathbf{b} \vee \mathbf{0}' = \mathbf{c}$. Finally note that $\mathbf{a} \vee \mathbf{b} \geqslant \mathbf{c}$, because if both a and b are available then considering the points where they differ c may be obtained. So we have $\mathbf{a}' = \mathbf{b}' = \mathbf{a} \vee \mathbf{b} = \mathbf{c}$, and a, b cannot have the same degree.

REMARK. The real a, b may also be chosen to be of minimal degree by using perfect trees in Q_3 instead of finite sequences.

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