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ON THE INERTIA GROUPS OF h-COBORDANT MANIFOLDS

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ABSTRACT. It is shown that if M and N are h-cobordant manifolds of dimension at least eight, then their special inertia groups are equal. (G. Brumfiel has shown that special inertia groups are not homotopy type invariants.) Examples are constructed for which the inertia group is an h-cobordism invariant.

Introduction. Recall that an exotic n-sphere is a smooth manifold homeomorphic to S^n. It was shown by J. Milnor [**15**] that such manifolds need not be diffeomorphic to S^n, hence the term "exotic spheres", and the group (under connected sum $\#$) of oriented exotic n-spheres is denoted Θ_n. Letting $\Gamma^n = \pi_0 \operatorname{Diff}(S^{n-1})$, the map $\Gamma^n \to \Theta_n$ given by $[f] \mapsto D^n \cup_f D^n = \Sigma(f)$ is an isomorphism for $n \geqslant 5$ [**15**]. The *inertia group* $I(M)$ of a smooth n-manifold M is defined as $\{\Sigma \in \Theta_n\colon \Sigma \# M \cong M\}$, and it is a consequence of the Disc Theorem of J. Cerf and R. Palais that $\Sigma(F) \in I(M)$ if and only if there is a diffeomorphism $F\colon M^n \backslash \operatorname{int} D^n \to M^n \backslash \operatorname{int} D^n$ satisfying $F \mid \partial D^n = f$. Thus $I(M)$ can be viewed as a measure of the stability of the differentiable structure of M under perturbations on an n-disc.

DEFINITION. Writing $M \# \Sigma(g)$ as $(M^n \backslash \operatorname{int} D^n) \cup_g D^n$, let $\iota\colon M \to M \# \Sigma(g)$ denote the PL homeomorphism defined by $\iota \mid M \backslash \operatorname{int} D^n =$ identity and $\iota \mid D^n = c(g)$ $=$ cone map on g. (The letter ι will be used for different M; which manifold is involved will be clear from the context.)

In [**4**] a *special inertia group* $I_h(M)$ is defined as $\{\Sigma \in \Theta_n$; there is a diffeomorphism $f\colon M \to M \# \Sigma$ homotopic to $\iota\}$.

R. DeSapio [**6**], K. Kawakubo [**13**] and R. Schultz [**17**] independently showed $I(M)$ is not a homotopy type invariant of M. They showed $I(S^p \times S^q) = 0$ and yet for some exotic spheres Σ^p and for some q, $I(\Sigma^p \times S^q) \neq 0$. Since Σ^p is PL homeomorphic to S^p, this also showed $I(M)$ is not a PL invariant of M.

In Proposition II.1 of [**4**] G. Brumfiel computes $I_h(M)$ for simply-connected M via D. Sullivan's theory of homotopy smoothings. In Remark II.12 Brumfiel shows there are manifolds P_1 and P_2, homotopy equivalent to $\mathbf{C}P(3)$, with $I_h(P_1 \times S^1) = 0$ and $I_h(P_2 \times S^1) = \mathbf{Z}_7$. Thus $I_h(M)$ is not a homotopy type invariant of M.

The goal of this paper is to show that $I_h(M)$ is an h-cobordism invariant for manifolds of dimension at least eight. Examples are constructed for which $I(M)$ is

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an h-cobordism invariant of M. It is a pleasure to acknowledge helpful conversations with Ian Hambleton, Terry Lawson, Youn Lee and Brian Mortimer. In addition, the referee's useful comments are appreciated.

2. Some special cases.

PROPOSITION 1. *Let* $(W^{n+1}; M, N)$ *be an h-cobordism with* $n \geqslant 5$ *and let* $\sigma = \tau(W, M) \in \mathrm{Wh}(\pi_1 M)$. *For* $\Sigma \in I(M)$, *if there is a diffeomorphism* $f\colon M \to M\#\Sigma$ *with* $f_*\sigma = \iota_*\sigma$, *then* $\Sigma \in I(N)$.

PROOF. Let V denote the connected sum along the cobordism of W with $\Sigma \times I$, so $(V; M\#\Sigma, N\#\Sigma)$ is an h-cobordism. Observe that $\tau(V, M\#\Sigma) = \iota_*\sigma \in \mathrm{Wh}(\pi_1 M\#\Sigma)$. Since $f_*\sigma = \iota_*\sigma$, it follows from the uniqueness theorem for h-cobordisms (11.3 of [**16**]) that f extends to a diffeomorphism $W \to V$, so $N \cong N\#\Sigma$.

COROLLARY 1. *If* $\pi_1 M$ *admits only inner automorphisms, then* $I(M) = I(N)$ *for any h-cobordism* $(W^{n+1}; M, N)$ *with* $n \geqslant 5$.

PROOF. For any diffeomorphism $f\colon M \to M\#\Sigma$, $(f^{-1}\iota)_\#$ induces an inner automorphism on $\pi_1 M$ and so, by 6.1 of [**16**], $f_* = \iota_*$ on $\mathrm{Wh}(\pi_1 M)$.

EXAMPLE 1. If $\pi_1 M$ is complete, then $I(M) = I(N)$. Z. Janko's sporadic finite simple group J_1 is complete [**12**] and by a theorem of H. Bass (6.2 of [**15**]) $\mathrm{Wh}(J_1)$ has rank five.

COROLLARY 2. *Let* $(W^{n+1}; M, N)$ *be an h-cobordism with* $n \geqslant 5$. *Then* $I_h(M) \subset I(N)$.

PROOF. If $\Sigma \in I_h(M)$, then $\iota \simeq f\colon M \to M\#\Sigma$ and so $f_* = \iota_*$ on Whitehead groups.

More work is required to show $\Sigma \in I_h(N)$ (see Corollary 2 of §5), but an attempt can be made to modify this argument and compare $I_h(M)$ with $I_h(N)$.

COROLLARY 3. *Let* $(W^{n+1}; M, N)$ *be an h-cobordism with* $n \geqslant 5$ *and odd, and suppose conjugation* (*see p.* 373 *of* [**16**]) *is trivial in* $\mathrm{Wh}(\pi_1 M)$. *Then* $I_h(M) = I_h(N)$.

PROOF. Take $\Sigma \in I_h(M)$ and $f\colon M \to M\#\Sigma$ homotopic to ι. As above, let V denote the connected sum along the cobordism of W with $\Sigma \times I$, and let $(Y; N, N\#\Sigma)$ denote the h-cobordism $\overline{W} \cup_f V$, where $(\overline{W}; N, M)$ is the reflection of $(W; M, N)$ (see 11.5 of [**16**]). Measuring torsions in the middle $\mathrm{Wh}(\pi_1 M)$, it follows that

$$\tau(Y, N) = \tau(\overline{W}, N) + f_*^{-1}\tau(V, M\#\Sigma) = \tau(\overline{W}, N) + f_*^{-1}\iota_*\tau(W, M)$$

$$= (-1)^n\bar{\tau}(W, M) + \tau(W, M) = -\tau(W, M) + \tau(W, M) = 0.$$

The s-cobordism theorem implies there is a trivialization $g\colon Y \to N \times I$. From g and the homotopy between f and ι, a homotopy between $(g|\partial_+ Y)^{-1}\colon N \to N\#\Sigma$ and ι can be constructed, so $\Sigma \in I_h(N)$.

EXAMPLE 2. Conjugation is trivial in $\mathrm{Wh}(\mathbf{Z}_q)$ (see Theorem 1 of [**1**] or Proposition 4.2 of [**3**]). If $(W; L_q, M)$ is an h-cobordism, L_q a $(2n+1)$-dimensional lens space, and $\tau(W, L_q) \neq 0$, then $L_q \not\cong M$ (Corollary 12.13 of [**16**]) and so lens spaces L_q provide examples of h-cobordisms which $L_q \not\cong M$ and $I_h(L_q) = I_h(M)$. (Examples

with nonzero torsion abound: recall $\mathrm{Wh}(\mathbf{Z}_q) \neq 0$ if $q \neq 1, 2, 3, 4, 6$). In many instances it is easy to see that $I_h(L_q)$—in fact, $I(L_q)$—cannot be all of Θ_n. A diffeomorphism $\Sigma \# L_q \to L_q$ lifts to a diffeomorphism of universal covers $q\Sigma \# S^{2n+1} \to S^{2n+1}$. Since $I(S^{2n+1}) = 0$, it follows that the order of Σ must divide q.

EXAMPLE 3. If $p \equiv 3 \pmod 4$, $k \geqslant 3$, and M is h-cobordant to $L_p^3 \times S^{2k}$, then Theorem 6.1 of [**10**] shows $L_p^3 \times S^{2k} \not\cong M$ and yet by Corollary 3 above, $I_h(M) = I_h(L_p^3 \times S^{2k})$. Since $I(S^3 \times S^{2k}) = 0$, an argument analogous to that given in Example 2 shows the order of Σ must divide p if $\Sigma \in I(L_p^3 \times S^{2k})$.

3. Inertia groups of connected sums. For all n-manifolds M and N, the Disc Theorem shows that a diffeomorphism $f\colon N \to N\#\Sigma$ extends to a diffeomorphism $f\#1\colon N\#M \to (N\#\Sigma)\#M$. In fact, one has $I(N) + I(M) \subset I(N\#M)$. To show that inclusion is sometimes strict, D. Wilkens [**18**] constructed 15-manifolds M and N with $I(M) = I(N) = 0$ and $I(M\#N) = \mathbf{Z}_{127}$.

As a model for the proof of the theorem (§5), the behavior of I_h with respect to connected sum will be studied.

PROPOSITION 2. *For all n-manifolds N and M, $n \geqslant 5$, $I_h(M) \subset I_h(N\#M)$.*

PROOF. Take $\Sigma \in I_h(M)$ and let $F\colon M \times I \to (M\#\Sigma) \times I$ be the track of a homotopy with F_1 a diffeomorphism and $F_0 = \iota$. Writing $M\#\Sigma$ as $(M \setminus \mathrm{int}\, D^n) \cup_f D^n$ and M as $(M \setminus \mathrm{int}\, D^n) \cup_{\mathrm{id}} D^n$, and selecting $x_0 \in M \setminus D^n$, the diffeomorphism F_1 can be isotoped to fix x_0. After modification by a homotopy, F can be taken to be smooth on a neighhorhood of $x_0 \times I$. Let p denote the projection $(M\#\Sigma) \times I \to (M\#\Sigma) \times \{1\}$. By transversality, $F|x_0 \times I$ and $pF|x_0 \times I$ can be taken to be smooth imbeddings. In $(M\#\Sigma) \times I$ the circle $(x_0 \times I) \cup F(x_0 \times I) \cup pF(x_0 \times I)$ bounds a D^2 (imbedded since $n \geqslant 5$), which can be used to pseudo-isotope $F|x_0 \times I$, relative to $x_0 \times 0$, to the inclusion. Then by Theorem 2.1 of [**11**] and isotopy extension, there is a diffeomorphism $H\colon (M\#\Sigma) \times I \to (M\#\Sigma) \times I$ satisfying $HF|x_0 \times I =$ inclusion and $H|(M\#\Sigma) \times 0 =$ identity.

Let $D(t)$ denote the derivative of HF at the point (x_0, t). Then $D(0)$ and $D(1)$ are invertible, so by the continuity of the determinant $D(t)$ is invertible for $0 \leqslant t < \varepsilon$ and $1 - \varepsilon < t \leqslant 1$. Modifying HF on an arbitrarily small neighborhood of $x_0 \times [\varepsilon, 1 - \varepsilon]$, $D(t)$ is invertible for all $t \in [0, 1]$. The inverse function theorem implies that for some small n-disc A around x_0, $HF|A \times I$ is an imbedding, hence provides a framing for $A \times I \subset (M\#\Sigma) \times I$ differing from the standard framing $T = D^n \times I \to \nu(x_0) \times I \subset (M\#\Sigma) \times I$ by a bundle diffeomorphism determined by $\lambda \in [(I, \partial I), (\mathrm{SO}(n), [\mathrm{id}])]$. Let $S\colon D^n \times I \to \nu(x_0) \times I \subset M \times I$ denote the standard framing. Then $T^{-1}HFS(x, t) = (\lambda(t) \cdot x, t)$, and the fact that $\lambda(0) = \lambda(1) = [\mathrm{id}]$ can be used to produce a self-diffeomorphsim E_1 of $D^n \times I$ satisfying $E_1|(\partial D^n \times I) \cup (1/2 D^n \times 0) =$ inclusion and $E_1|1/2 D^n \times I = T^{-1}HFS|1/2 D^n \times I$. (Specifically, let $\alpha = [0, 1] \to [0, 1]$ be a smooth map satisfying $\alpha(s) = 1$ if $0 \leqslant s \leqslant \frac{1}{2}$ and $\alpha(1) = 0$. Then E_1 is given by $E_1(x, t) = (\lambda(\alpha(\|x\|)t) \cdot x, t)$.) Thus TE_1T^{-1} extends by the identity to a diffeomorphism E_2 of $(M\#\Sigma) \times I$. The composition $E_2^{-1}HF$ is the identity along $A \times I$, hence extends by the identity to a homotopy between a diffeomorphism $M\#N \to (M\#\Sigma)\#N$ and ι. This completes the proof.

4. Stably diffeomorphic manifolds. One method of studying h-cobordant manifolds is the following well-known result (see the proof of the Proposition of [**8**]).

LEMMA 1. *If M^n is h-cobordant to N^n and $n \geqslant 5$, then for sufficiently large m there is a diffeomorphism $M\#m(S^2 \times S^{n-2}) \cong N\#m(S^2 \times S^{n-2})$.*

PROOF. Any h-cobordism between M and N can be constructed by trivially attaching m 2-handles to $M \times I$ and then attaching m 3-handles to obtain the correct torsion (see Theorem 11.1 of [**16**]). Since the 2-handles are attached trivially, a level surface between the 2-handles and the 3-handles is $M\#m(S^2 \times S^{n-2})$. In the dual handlebody decomposition of the h-cobordism, one starts with $N \times I$, attaches m $(n-2)$-handles trivially, and then attaches m $(n-1)$-handles to obtain the correct torsion. A level surface between the $(n-2)$-handles and the $(n-1)$-handles is $N\#m(S^2 \times S^{n-2})$. This completes the proof.

Let M and N be h-cobordant n-manifolds, $n \geqslant 5$, and let g: $M\#m(S^2 \times S^{n-2}) \to N\#m(S^2 \times S^{n-2})$ be the diffeomorphism given by Lemma 1. If $\Sigma \in I(N)$, there is a diffeomorphism h: $N \to N\#\Sigma$ and the composition f given by

$$M\#m(S^2 \times S^{n-2}) \xrightarrow{g} N\#m(S^2 \times S^{n-2})$$

$$\xrightarrow{h\#1} \Sigma\#N\#m(S^2 \times S^{n-2}) \xrightarrow{1\#g^{-1}} \Sigma\#M\#m(S^2 \times S^{n-2})$$

shows $\Sigma \in I(M\#m(S^2 \times S^{n-2}))$. The argument to show $\Sigma \in I(M)$ proceeds by modifying f so successive summands of $S^2 \times S^{n-2}$ can be removed by surgery on S^2.

LEMMA 2. *If there is a diffeomorphism f: $M_0 = M^n\#(S^2 \times S^{n-2}) \to N^n\#(S^2 \times S^{n-2}) = N_0$ satisfying $f|S^2 \simeq$ inclusion for $S^2 = S^2 \times \{*\} \subset S^2 \times S^{n-2}$, and if $n \geqslant 7$, then there is a diffeomorphism $M \to N$.*

PROOF. The result will follow by a surgery on $S^2 \times D^{n-2} \subset S^2 \times S^{n-2}$ if it can be shown that f fixed this $S^2 \times D^{n-2}$. By transversality $f(S^2)$ can be taken to miss S^2 and the homotopy $S^2 \times I \to N_0$ between $f|S^2$ and the inclusion can be taken to be an imbedding (since $n \geqslant 7$), hence a pseudo-isotopy. Theorem 2.1 of [**11**] implies $f|S^2$ is isotopic to the inclusion, so by isotopy extension there is a diffeomorphism f': $M_0 \to N_0$ with $f'|S^2 \simeq$ inclusion.

Let ν_M denote the normal bundle $\nu(S^2 \to M_0)$ and let ν_N denote $\nu(S^2 \to N_0)$. Then by the tubular neighborhood theorem f' can be isotoped to f'' satisfying $f''(\nu_M) = \nu_N$ and $f''|\nu_M$ is a vector bundle isomorphism. Since $\pi_2 \mathrm{SO}(n-2) = 0$, f'' can be isotoped to give the standard identification on a neighborhood $S^2 \times D^{n-2}$ of S^2. Surgery on this framed S^2 gives the result.

COROLLARY 1. *If M^n is 2-connected, $n \geqslant 7$, and if there is a diffeomorphism f: $M\#m(S^2 \times S^{n-2}) \to N\#m(S^2 \times S^{n-2})$, then there is a diffeomorphism $M \cong N$.*

PROOF. The result follows from m applications of Lemma 2, provided there is a diffeomorphism h of $N\#m(S^2 \times S^{n-2})$ with $hf| + mS^2 \simeq$ inclusion. Since $\pi_2(N\#m(S^2 \times S^{n-2})) \cong \oplus m\langle S^2 \rangle \cong \mathbf{Z}^m$, it suffices to show any element of

$\mathrm{Aut}(\mathbf{Z}^m) = \mathrm{GL}(m, \mathbf{Z})$ can be induced by a diffeomorphism of $\#m(S^2 \times S^{n-2})$. Any element of $\mathrm{GL}(m, \mathbf{Z})$ can be written as a product of permutation matrices and "elementary" matrices (the identity matrix with ± 1 in one off-diagonal location). It suffices to give a diffeomorphism g of $(S_1^2 \times S^{n-2})\#(S_2^2 \times S^{n-2})$, inducing the automorphism $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ on π_2. Split $(S_1^2 \times S^{n-2})\#(S_2^2 \times S^{n-2})$ as $A \cup_1 A$, where $A = S_1^2 \times D^{n-2} \amalg S_2^2 \times D^{n-2}$, and observe that $A \cup_1 A$ admits an involution t interchanging the copies of A and fixing ∂A pointwise. Construct a diffeomorphism h: $A \to A$ as follows. Send disjoint neighborhoods of S_1^2 and S_2^2 to disjoint neighborhoods of $S_1^2 + S_2^2$ and S_2^2, respectively. Connect each pair of neighborhoods by a neighborhood of an arc joining S_1^2 and S_2^2. Then the remainder of the domain is $\partial A \times I$, as is the remainder of the range (by the h-cobordism theorem). Extend the given map to produce h, and extend h to the second copy of A by tht. This yields g.

COROLLARY 2. *If M^n is 2-connected and $n \geqslant 7$, then for all m,*

$$I(M) = I\big(M\#m(S^2 \times S^{n-2})\big).$$

PROOF. Suppose g: $(\Sigma\#M)\#m(S^2 \times S^{n-2}) \to M\#m(S^2 \times S^{n-2})$. Then Corollary 1 implies there is a diffeomorphism $\Sigma\#M \to M$, so $I(M\#m(S^2 \times S^{n-2})) \subset I(M)$. The reverse inclusion follows from the remark at the beginning of §3.

The connectivity assumptions of Corollaries 1 and 2 are necessary in order to homotope $f(+mS^2)$ into $\#m(S^2 \times S^{n-2})$ because otherwise $f(+mS^2) \cap M$ may have annular components which are nontrivial in $\pi_1(M \setminus \mathrm{int}\, D^n, \partial)$ or disc components which are nontrivial in $\pi_2(M \setminus \mathrm{int}\, D^n, \partial)$. In some specific instances, however, the connectivity hypotheses can be removed.

EXAMPLE 4. Let $(W, L_p^3 \times S^{2k}, N)$ be an h-cobordism with nontrivial torsion and let $p \equiv 3 \pmod 4$ and $k \geqslant 3$. By Theorem 6.1 of **[10]**, $N \not\cong L_p^3 \times S^{2k}$. In Example 3 it was shown that $I_h(N) = I_h(L_p^3 \times S^{2k})$. Lemma 2 can be used to show $I(N) = I(L_p^3 \times S^{2k})$. Given a diffeomorphism h: $L_p^3 \times S^{2k} \to \Sigma\#(L_p^3 \times S^{2k})$, construct

$$f = (1\#g)(h\#1)g^{-1}\colon N\#m(S^2 \times S^{2k+1}) \to \Sigma\#N\#m(S^2 \times S^{2k+1})$$

as above. Since $\pi_2((L_p^3 \times S^{2k})\#m(S^2 \times S^{2k+1})) = \oplus m\mathbf{Z}[\mathbf{Z}_p]$, generated by the inclusions of the m factors of S^2, $h\#1$ induces the identity on π_2. Consequently, $\iota^{-1}f$ also induces the identity on π_2, and m applications of Lemma 2 imply $I(L_p^3 \times S^{2k}) \subset I(N)$. The reverse inclusion is given by a similar argument.

EXAMPLE 5. In Example 2 it was shown that $I_h(L) = I_h(N)$ if N^{2n+1} is h-cobordant to a lens space L. Since $\pi_2(L\#m(S^2 \times S^{2n-1})) \cong \oplus m\mathbf{Z}[\mathbf{Z}_p]$, the argument of Example 4 can be rephrased to show $I(N) = I(L)$.

6. Stable diffeomorphisms homotopic to ι. The comparison of special inertia groups of h-cobordant manifolds is made by the following result.

THEOREM. *If a diffeomorphism f: $M\#(S^2 \times S^{n-2}) \to \Sigma\#M\#(S^2 \times S^{n-2})$ satisfies $f \simeq \iota$, and if $n \geqslant 8$, there is a diffeomorphism g: $M \to \Sigma\#M$ satisfying $g \simeq \iota$.*

PROOF. Let $S_1^2 = S^2 \times \{*\} \subset S^2 \times S^{n-2}$ in the domain of f and let $S_2^2 = S^2 \times \{*\}$ in the range of f. Isotope f so that $f(S_1^2)$ misses S_2^2. Since $\iota \mid S_1^2 =$ inclusion, the proof

proceeds by applying Lemma 2 to produce a diffeomorphism $f'\colon M \to M\#\Sigma$ and then extending the argument of Proposition 2 to show $f' \cong \iota$. Let $F\colon A \times I \to B \times I$ be the track of a homotopy with $F_1 = f$ and $F_0 = \iota$, and take F to be smooth near $S_1^2 \times I$. By transversality F can be taken to be an imbedding on $S_1^2 \times I$. Then Theorem 2.1 of [**11**] and isotopy extension imply there is an isotopy $F'\colon A \times [1,2] \to B \times [1,2]$ with $F_1' = f$, $F_2' = f'$, $f' \mid S_1^2 = \iota \mid S_1^2$, and $F' \mid S_1^2 \times \{t\} = F \mid S_1^2 \times \{2-t\}$. Let $G\colon A \times I \to B \times I$ be the homotopy given by $G(a,t) = F(a,2t)$ for $0 \leqslant t \leqslant \frac{1}{2}$ and $G(a,t) = F'(a,2t)$ for $\frac{1}{2} \leqslant t \leqslant 1$. Now $G \mid S_1^2 \times I$ is homotopic, rel $S_1^2 \times \{0,1\}$, to the inclusion, so by transversality (since $n \geqslant 8$) $G \mid S_1^2 \times I$ is pseudo-isotopic to the inclusion. Theorem 2.1 of [**11**] and isotopy extension give rise to a level-preserving diffeomorphism $H\colon B \times I \to B \times I$ satisfying $HF \mid S_1^2 \times I =$ inclusion and $H \mid B \times 0 =$ identity.

It remains to modify HF to be the inclusion on a neighborhood of $S_1^2 \times I$. Observe ι is the inclusion on a neighborhood of $S_1^2 \times 0$ and since $\pi_2 \mathrm{SO}(n-2) = 0$, $(HF)_1 = g$ can be taken to be the inclusion on a neighborhood of $S_1^2 \times 1$. Let $D^{n-2} \times S_i^2 \times I$ denote the standard framing of $\nu(S_i^2 \times I \to A \times I)$. Recall HF is the track of a homotopy—that is, HF preserves the I-coordinate. Modify HF on $D^{n-2} \times S_1^2 \times I$, rel $\partial D^{n-2} \times S_1^2 \times I$, in two stages. First, send (x, y, t) to $(0, y, t)$ if $\|x\| \leqslant \frac{1}{2}$ and to $(x(2\|x\| - 1), y, t)$ if $\frac{1}{2} \leqslant \|x\| \leqslant 1$. A map $D^{n-2} \times S_1^2 \times I \to B \times I$ which agrees with HF on $\partial D^{n-2} \times S_1^2 \times I$ and which is the inclusion on $1/2 D^{n-2} \times S_1^2 \times I$ similarly can be shrunk to send $1/2 D^{n-2} \times S_1^2 \times I$ to $0 \times S_1^2 \times I$. The second modificiation of HF is the reverse of this homotopy. Thus modified, HF provides a homotopy between ι and a diffeomorphism g, and HF restricts to the inclusion on $D^{n-2} \times S_1^2 \times I$. Using this framing, surgery on $S_1^2 \times I$ yields the result.

COROLLARY 1. *If $n \geqslant 8$ then $I_h(M^n) = I_h(M^n \# m(S^2 \times S^{n-2}))$ for all m.*

PROOF. Proposition 2 implies $I_h(M) \subset I_h(M \# m(S^2 \times S^{n-2}))$. The reverse inclusion follows from m applications of the theorem.

COROLLARY 2. *If M^n is h-cobordant to N^n and $n \geqslant 8$, then $I_h(M) = I_h(N)$.*

PROOF. Lemma 1 implies that for sufficiently large m, $I_h(M \# m(S^2 \times S^{n-2})) = I_h(N \# m(S^2 \times S^{n-2}))$. Corollary 1 implies $I_h(M) = I_h(N)$.

In a sequel this argument will be extended to study inertial h-cobordisms and diffeomorphism groups of h-cobordant manifolds.

This corollary shows that the pathology detected by Brumfiel in proving $I_h(M)$ is not a homotopy type invariant does not persist if "homotopy equivalent" is replaced by "h-cobordant".

EXAMPLE 6. In [7] F. Farrell and W.-C. Hsiang construct, for each $n \geqslant 6$, h-cobordisms $(W^{n+1}; M, N)$ which are noninertial; that is, M is not diffeomorphic to N. (In the examples of Farrell and Hsiang, M and N are not even homeomorphic.) If $n \geqslant 8$, Corollary 2 implies $I_h(M) = I_h(N)$. In this example $M = L^3 \times T^{n-3}$ where L^3 is a certain lens space and T^{n-3} is the $(n-3)$-dimensional torus.

To make a nontrivial application of Corollary 2, there must be noninertial h-cobordisms from M. It should be pointed out that there are many examples of

inertial h-cobordisms. One result along these lines is given by T. Lawson in Corollary 4 of [**14**]; if n is even, q is odd, and $\pi_1 M^n = \mathbf{Z}_q$, then every h-cobordism from M is inertial. In fact Proposition 1 of [**14**] shows that every h-cobordism from an even-dimensional manifold M^n is inertial if conjugation is trivial in $\mathrm{Wh}(\pi_1 M)$ and $L^h_{n+1}(\pi_1 M) = 0$. It follows from Theorem 1 of [**1**] or Proposition 4.2 of [**3**], and from Theorem 1 of [**2**] that these conditions hold if $\pi_1 M$ is torsion abelian of odd order. In every dimension exceeding four, Corollaries 1.2 of [**9**] and 2 of [**14**] construct manifolds from which every h-cobordism is inertial. The Proposition of [**8**] shows that for every manifold M^n, $n \geqslant 5$, every h-cobordism is inertial after stabilizing M by taking connected sum with a sufficient number of copies of $S^p \times S^{n-p}$, $2 \leqslant p \leqslant n-2$. On the other hand, Theorem 6.1 and Lemma 6.5 of [**10**] construct examples of noninertial h-cobordisms.

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