

BRANCHED CYCLIC COVERS OF SIMPLE KNOTS¹

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ABSTRACT. The Blanchfield pairing of simple $(2q - 1)$ -knots, $q \geq 2$, is used to give an algebraic characterization of those knots which may arise as m -fold branched cyclic covers of simple knots.

0. Introduction. In this paper we use the classification of odd-dimensional simple knots in terms of their Blanchfield pairing in **[K and T]** to give an algebraic characterization of those knots which can arise as branched cyclic covers of other knots.

We work in the piecewise linear category throughout, where all embeddings and isotopies are taken to be locally flat. An n -knot is an embedding $S^n \hookrightarrow S^{n+2}$, where the spheres are oriented; two such knots k, l are equivalent if there is an isomorphism of the pairs $(S^{n+2}, kS^n), (S^{n+2}, lS^n)$.

I would like to express my thanks to Cherry Kearton for suggesting this line of research, and for pointing out a direct proof for Theorem 1.

1. Suppose we have an odd-dimensional simple knot $k = (S^{2q+1}, S^{2q-1})$, $q \geq 2$, with Alexander module A_t and Blanchfield pairing $\langle \cdot, \cdot \rangle_t$, the t subscript denoting a generator for the group of covering translations of \tilde{X} , the infinite cyclic cover of the exterior X of the knot. We wish to determine conditions on the module and pairing which will allow k to arise as the m -fold branched cyclic cover of another knot l . If k does so arise, then \tilde{X} will also be the infinite cyclic cover of l ; but the group of covering translations will now be generated by a homeomorphism u such that $u^m = t$. This homeomorphism will allow us to consider A_t as a $\mathbf{Z}[u, u^{-1}]$ -module A_u , which will be the Alexander module of l . It will also enable us to write down the Blanchfield pairing $\langle \cdot, \cdot \rangle_u$ of l as follows.

Take two elements a, b of $A_t = H_q(\tilde{X})$. A_t is a Λ -torsion module ($\Lambda = \mathbf{Z}[t, t^{-1}]$), so there exists a nonzero $\pi(t) \in \Lambda$ such that $\pi a = 0$. Choose a triangulation of \tilde{X} induced by a triangulation of the exterior of l , and let C_q be the group of q -chains, \tilde{C}_{q+1} the group of $(q+1)$ -chains in the dual triangulation. As $\pi a = 0$, we may choose an element α of \tilde{C}_{q+1} whose boundary represents πa ; we also choose $\beta \in C_q$ representing b . Then we define **[B, K]**

$$\langle a, b \rangle_t = \left[\left(\sum_{i=-\infty}^{\infty} I(\alpha, t^i \beta) t^i \right) / \pi(t) \right] \in (Q(t) / \mathbf{Z}[t, t^{-1}]),$$

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and

$$\langle a, b \rangle_u = \left[\left(\sum_{i=-\infty}^{\infty} I(\alpha, u^i \beta) u^i \right) / \pi(u^m) \right] \in (Q(u)/\mathbf{Z}[u, u^{-1}]).$$

We now group the infinite sum into powers of u having the same value modulo m . Let M be a complete set of representatives of the integers modulo m , for instance $\{0, 1, \dots, m-1\}$. Then we have

$$\begin{aligned} \langle a, b \rangle_u &= \frac{\sum_{k \in M} u^k \sum_{i=-\infty}^{\infty} I(\alpha, u^{im+k} \beta) u^{im}}{\pi(u^m)} \\ &= \sum_{k \in M} u^k \left[\frac{\sum_{i=-\infty}^{\infty} I(\alpha, t^i (u^k \beta)) (u^m)^i}{\pi(u^m)} \right] \\ &= \sum_{k \in M} u^k \cdot \mu \langle a, u^k b \rangle_t \end{aligned}$$

where

$$\mu: \frac{Q(t)}{\mathbf{Z}[t, t^{-1}]} \rightarrow \frac{Q(u)}{Q[u, u^{-1}]}$$

is $f(t) \mapsto f(u^m)$.

2. Given that $(A_t, \langle \cdot, \cdot \rangle_t)$ is the module and pairing of such a simple knot, we know it must satisfy the Levine axioms:

(L1) A_t is a finitely generated Λ -torsion-module.

(L2) Multiplication by $(1-t)$ acts as an automorphism of A_t .

(L3) $\langle \cdot, \cdot \rangle_t$ is $e = (-1)^{q+1}$ -Hermitian, that is

$$\langle a, b \rangle_t = e \overline{\langle b, a \rangle_t} \quad \text{and} \quad t \langle a, b \rangle_t = \langle ta, b \rangle_t$$

where $\overline{}$ denotes the involution of $Q(t)/\mathbf{Z}[t, t^{-1}]$ defined by $\overline{f(t)} = f(t^{-1})$.

(L4) $\langle \cdot, \cdot \rangle_t$ is nonsingular, that is, the adjoint map:

$$A_t \rightarrow \text{Hom}_{\Lambda}(A_t, Q(t)/\mathbf{Z}[t, t^{-1}]), \quad a \mapsto (x \mapsto \langle x, a \rangle_t)$$

is a (conjugate-linear) isomorphism.

Furthermore [K, T], these conditions characterise the modules and pairings which can arise, together with the condition that the signature of the corresponding quadratic form must be divisible by 16 for $q = 2$; that is, any such module and pairing arises as that of a simple knot; and if two knots have isometric pairings, they are equivalent.

To derive the algebraic condition for k to be the m -fold branched cyclic cover of a knot, we shall need to use the following trick repeatedly.

LEMMA. Suppose M is a set of integers having distinct values modulo m and we are given that

$$\frac{\sum_{k \in M} \sum_{i=-\infty}^{\infty} a_{k+im} u^{k+im}}{\pi(u^m)} = 0 \quad \text{in} \quad \frac{Q(u)}{\mathbf{Z}[u, u^{-1}]}.$$

Then, for all $k \in M$,

$$\frac{\sum_{i=-\infty}^{\infty} a_{k+im} t^i}{\pi(t)} = 0 \quad \text{in } \frac{\mathbf{Q}(t)}{\mathbf{Z}[t, t^{-1}]}.$$

PROOF. Since

$$\pi(u^m) \left| \sum_{i=-\infty}^{\infty} b_i u^i \right| \Leftrightarrow \pi(u^m) \left| \sum_{i=k \pmod{m}} b_i u^i \right| \quad \forall k \in \mathbf{Z}.$$

If the module A_u and the pairing $\langle \cdot, \cdot \rangle_u$ defined above satisfy the Levine conditions, we may construct a knot l corresponding to them. Let k' be the m -fold branched cyclic cover of l , which is a sphere because $1 - u^m$ is an automorphism of A_u . The Alexander module of k' is clearly A_t , and we have

$$\sum_{k=0}^{m-1} u^k \mu \langle a, u^k b \rangle'_t = \langle a, b \rangle_u = \sum_{k=0}^{m-1} u^k \mu \langle a, u^k b \rangle_t,$$

where $\langle \cdot, \cdot \rangle'_t$ is the Blanchfield pairing of k' . Using the lemma for $k \equiv 0 \pmod{m}$ we see that $\langle a, b \rangle_t = \langle a, b \rangle'_t$; so k and k' are equivalent, and k does arise as an m -fold branched cyclic cover. Then we have

THEOREM 1. $(A_u, \langle \cdot, \cdot \rangle_u)$ satisfies the Levine conditions (and hence k is an m -fold branched cyclic cover) if and only if u is an isometry of $(A_t, \langle \cdot, \cdot \rangle_t)$ with $u^m = t$.

PROOF. First the “only if” part: For $\langle \cdot, \cdot \rangle_u$ to satisfy (L3) we must have

$$\begin{aligned} \langle a, b \rangle_u &= e \overline{\langle b, a \rangle_u} \quad \forall a, b \in A_u \\ &\Leftrightarrow \sum_{k=0}^{m-1} u^k \mu \langle a, u^k b \rangle_t = e \sum_{k=0}^{m-1} \overline{u^k \mu \langle b, u^k a \rangle_t} \\ &= \sum_{k=0}^{m-1} u^{-k} \mu \langle u^k a, b \rangle_t \quad \text{as } \langle \cdot, \cdot \rangle_t \text{ is } e\text{-Hermitian} \\ &\Leftrightarrow \langle a, u^k b \rangle_t = t^{-1} \langle u^{m-k} a, b \rangle_t \quad \forall 0 \leq k < m, \text{ by the lemma} \\ &\Leftrightarrow \langle a, u^k b \rangle_t = \langle u^{-k} a, b \rangle_t \quad \forall 0 \leq k < m \\ &\Leftrightarrow u \text{ is an isometry of } \langle \cdot, \cdot \rangle_t. \end{aligned}$$

Conversely, A_u is clearly finitely generated (L1), and $(1 - u)$ is an automorphism since

$$(1 - u)(1 + u + \cdots + u^{m-1}) = 1 - u^m = 1 - t.$$

So we must prove (L3) and (L4) under the assumption that u is an isometry. We have already proved the first half of (L3) above; for the second half we have

$$\begin{aligned}\langle ua, b \rangle_u &= \sum_{k=0}^{m-1} u^k \mu \langle ua, u^k b \rangle_t \\ &= \sum_{k=0}^{m-1} u^k \mu \langle a, u^{k-1} b \rangle_t \quad \text{as } u \text{ is an isometry} \\ &= u \sum_{k=-1}^{m-2} u^k \mu \langle a, u^k b \rangle_t = u \langle a, b \rangle_u.\end{aligned}$$

Finally we prove (L4) in two parts.

(i) *The adjoint map is injective.* Suppose $\langle a, b \rangle_u = 0 \quad \forall a \in A_u$. Then

$$\sum_{k=0}^{m-1} u^k \cdot \mu \langle a, u^k b \rangle_t = 0 \quad \forall a \in A_t.$$

We may use the lemma for $k = 0$; whence

$$\langle a, b \rangle_t = 0 \quad \forall a \in A_t$$

$\therefore b = 0$ as required, since $\langle \cdot, \cdot \rangle_t$ is nonsingular.

(ii) *The adjoint map is surjective.* Suppose $a \mapsto f_a$ is a u -linear map

$$A_u \rightarrow \mathbf{Q}(u)/\mathbf{Z}[u, u^{-1}].$$

If $\Delta(t)$ is the Alexander polynomial of k , then we may write

$$f_a = \frac{\sum_{k=0}^{m-1} u^k f_a^{(k)}(u^m)}{\Delta(u^m)}.$$

Then $a \mapsto f_a^{(0)}(t)/\Delta(t)$ is a t -linear map

$$A_t \rightarrow \mathbf{Q}(t)/\mathbf{Z}[t, t^{-1}].$$

So, as $\langle \cdot, \cdot \rangle_t$ is nonsingular, there is an x in A_t such that

$$\langle a, x \rangle_t = f_a^{(0)}(t)/\Delta(t).$$

Then we have

$$\begin{aligned}\langle a, x \rangle_u &= \sum_{k=0}^{m-1} u^k \mu \langle a, u^k x \rangle_t \\ &= \sum_{k=0}^{m-1} u^k \mu \langle u^{-k} a, x \rangle_t \quad \text{as } u \text{ is an isometry} \\ &= \left(\sum_{k=0}^{m-1} u^k f_{u^{-k}a}^{(0)}(u^m) \right) / \Delta(u^m) \\ &= \left(\sum_{k=0}^{m-1} u^k f_a^{(k)}(u^m) \right) / \Delta(u^m) \quad \text{as } a \mapsto f_a \text{ is } u\text{-linear} \\ &= f_a \quad \text{as required.}\end{aligned}$$

We have proved that \langle , \rangle_u is a genuine Blanchfield pairing, and as we have seen this suffices to demonstrate the theorem for $n > 2$. Suppose now that \langle , \rangle_t is the Blanchfield pairing of a 3-knot. As the Levine conditions depend only on the value of q modulo two the above proof shows that \langle , \rangle_u is the Blanchfield pairing of a 7-knot l_7 whose m -fold cover k_7 has pairing \langle , \rangle_t . Any lift of a Seifert surface of l_7 to the complement of k_7 will then be a Seifert surface for k_7 ; thus the quadratic pairings corresponding to \langle , \rangle_t and \langle , \rangle_u have the same signature, namely that of the surface; so it follows that if \langle , \rangle_t is the Blanchfield pairing of a 3-knot, so is \langle , \rangle_u , completing the proof.

Finally, we prove a condition for two odd-dimensional simple knots having k as an m -fold branched cyclic cover to be equivalent.

THEOREM 2. *Suppose u, v are two isometries of $(A_t, \langle , \rangle_t)$ with $u^m = t = v^m$. Then $(A_u, \langle , \rangle_u)$ and $(A_v, \langle , \rangle_v)$ are isometric, and hence correspond to equivalent knots, if and only if u and v are conjugate by an isometry of $(A_t, \langle , \rangle_t)$.*

PROOF. We write both \langle , \rangle_u and \langle , \rangle_v with values in $\mathbf{Q}(u)/\mathbf{Z}[u, u^{-1}]$. Then $\phi: (A_u, \langle , \rangle_u) \rightarrow (A_v, \langle , \rangle_v)$ is an isometry

$$\begin{aligned} \Leftrightarrow \langle a, b \rangle_u &= \langle \phi a, \phi b \rangle_v \quad \forall a, b \in A_u \\ \Leftrightarrow \sum_{k=0}^{m-1} u^k \mu \langle a, u^k b \rangle_t &= \sum_{k=0}^{m-1} u^k \mu \langle \phi a, v^k \phi b \rangle_t \quad \forall a, b \in A_u \\ \Leftrightarrow \langle a, u^k b \rangle_t &= \langle \phi a, v^k \phi b \rangle_t \quad \forall a, b \in A_u, 0 \leq k < m \text{ by the lemma} \\ \Rightarrow \phi &\text{ is an isometry of } \langle , \rangle_t \quad (k = 0), \end{aligned}$$

and taking $k = 1$ we see that $u = \phi^{-1}v\phi$, since \langle , \rangle_t is nonsingular.

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