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ON FIXED RINGS OF AUTOMORPHISMS

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ABSTRACT. In this note we present simple proofs of some well-known results on fixed rings of finite automorphism groups of associative rings.

In this note G denotes a finite group acting as automorphisms on an associative ring R, $R^G = \{x \in R \mid x^g = x \text{ for all } g \in G\}$ and $t\colon R \to R^G$ is the trace map, i.e. $t(x) = \sum_{g \in G} x^g$ for any $x \in R$. By $|G|$ we denote the order of G and by J the Jacobson radical.

In this note we present elementary proofs of the following important results.

THEOREM A (BERGMAN AND ISAACS [1]). *There exists a positive integer* $n(G)$ *(depending only on* G*) such that if* $r(R) = \{x \in R \mid t(R)x = 0\}$ *then* $(|G|r(R))^{n(G)} = 0$.

THEOREM B (MARTINDALE [3]). $|G|J(R^G) \subseteq J(R)$.

REMARK 1. (a) Bergman and Isaacs proved that

$$n(G) \leqslant \prod_{m=1}^{|G|} \left(\binom{|G|}{m} + 1 \right).$$

(b) Simple induction arguments applied to Theorem A show that if R^1 is the usual extension of R to a ring with unity, then for any positive integer d,

$$(|G|R)^{(n(G)+1)^d} \subseteq |G|R^1 t(R)^d R$$

(cf. [1, Lemma 2.2]).

We need the following results.

I [2, Proposition 3.6.1]. The intersection of a finite number of modular maximal right ideals of a ring A is a modular right ideal of A.

II [4]. If the polynomial ring $A(X, x) = (A\{\{X\}\})[x]$ in an indeterminate x over the power series ring $A\{\{X\}\}$ in a set X of at least two noncommutative indeterminates is J-radical, then the ring A is nilpotent.

REMARK 2. Result II is particularly easy if card $X =$ card A. Indeed, if $A(X, x) \in J$ then $A\{\{X\}\}$ is nil (cf. [2]). Since card $X =$ card A we can label the indeterminates in X as $\{x_r \mid r \in R\}$. Now for some n, $(\sum_{r \in A} r x_r)^n = 0$. This implies $A^n = 0$.

LEMMA 1. *If* M *is a maximal modular right ideal of* R *then for some* $e \in R$ *and all* $a \in R$, $t(e)a - |G|a \in M$.

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PROOF. By I, $\overline{M} = \bigcap_{g\in G} M^g$ is a modular right ideal of R, so for some $e \in R$ and all $x \in R$, $ex - x \in \overline{M}$. Since $\overline{M}$ is G-invariant we conclude that $e^g a - a \in \overline{M}$ for $g \in G$ and $a \in R$. Consequently, $t(e)a - |G|a \in \overline{M} \subseteq M$ for $a \in R$.

PROOF OF THEOREM A. If the statement does not hold then for any positive integer n there exists a ring R_n such that $(|G|r(R_n))^n \neq 0$. Let $R = \oplus R_n$ be the discrete direct sum of rings R_n. Extend G to act on R componentwise. Then $r(R) = \oplus r(R_n)$, so $|G|r(R)$ is not nilpotent. Now extending the action of G from R to $R(X, x)$ in the natural way we obtain $r(R)(X, x) = r(R(X, x))$. By Lemma 1,

$$|G|r(R)(X, x) = |G|r(R(X, x)) \subseteq J(R(X, x)).$$

Thus II implies $|G|r(R)$ is nilpotent, a contradiction.

LEMMA 2. *If $t(R) \in J$ then $|G|R \in J$.*

PROOF. If $|G|R \notin J$ then $|G|R$ contains a maximal modular right ideal M. Applying Lemma 1 to $|G|R$ we obtain that for some $e \in R$ and all $a \in R$, $|G|^2(t(e)a - a) \in M$. Now treating $t(e)a - a$ as elements of R^1 we obtain $|G|^2(t(e) - 1)R \subseteq M$. Since $t(R) \in J$, $t(e) - 1$ is invertible in R^1. This and the fact that R is an ideal of R^1 imply $(t(e) - 1)R = R$. Consequently, $|G|^2 R \subseteq M$. This is impossible as $M/|G|^2R$ is a maximal modular right ideal of $A = |G|R/|G|^2R$ and $A^2 = 0$.

REMARK 3. Lemma 2 applied to $R(X, x)$ and II imply that if $t(R)$ is nilpotent then so is $|G|R$ (cf. [**1**, Proposition 2.3]). This result is also a consequence of Remark 1(b).

PROOF OF THEOREM B. Let $R_1 = J(R^G)R^1$. It is clear that R_1 is a G-invariant right ideal of R and $t(R_1) \subseteq J(R^G)$. Since $t(R_1)$ is an ideal of R^G, $t(R_1) \in J$. Hence by Lemma 2, $|G|R_1 \subseteq J(R)$. Consequently, $|G|J(R^G) \subseteq J(R)$.

REMARK 4. Let P and L denote the prime and the locally nilpotent radicals, respectively. It is known [**4**] that $J(R(X, x)) \neq 0$ if and only if $P(R) \neq 0$ and that $J(R\{X\}) = L(R)\{X\}$, where $R\{X\}$ is the polynomial ring in a set X of at least two noncommutative indeterminates over R. This, along with II and the respective results concerning J imply

(a) if $t(R) \in P$ ($t(R) \in L$) then $|G|R \in P$ ($|G|R \in L$),

(b) $|G|P(R^G) \subseteq P(R)$ and $|G|L(R^G) \subseteq L(R)$.

Consequently, if R contains no $|G|$-torsion then $P(R^G) = P(R) \cap R^G$ and $L(R^G) = L(R) \cap R^G$.

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