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RECOVERY OF H^p-FUNCTIONS

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ABSTRACT. Let there be given finitely many points $\{\alpha_k\}_1^n$ from the unit disc. If f is a H^p-function then how well can the value of f at $z = 0$ be approximated by linear means $\sum_1^n c_k f(\alpha_k)$? We give the optimal constants c_k and get, as a corollary, the possibility of the approximation of f by operators of the form $\sum_1^n f(\alpha_k)p_k$ with polynomials p_k. The order of approximation depends on the distance $\sum_1^n(1 - |\alpha_k|)$ of the point system from the unit circle.

Let $U = \{z \mid |z| < 1\}$ be the unit disc, and H^p $(1 \leqslant p < \infty)$ and A the usual Hardy space and disc algebra, respectively. In what follows, H will denote either of these spaces with corresponding norm $\|\cdot\|_H$. Let us consider the following problem: if $\{\alpha_k\}_1^\infty \subseteq U$ is a sequence of points, $f \in H$, and we know the values $f(\alpha_k)$, then how can the value of f at another point α be determined, or, more generally, how can we represent Lf for a fixed, but otherwise arbitrary, linear functional $L \in H^*$ via the values $\{f(\alpha_k)\}_1^\infty$? We shall consider a somewhat more general situation, namely when we are given a point system $\{\alpha_{nk}\}_{n=1,1\leqslant k\leqslant n}^\infty$ and we know at the nth step only the values $\{f(\alpha_{nk})\}_{k=1}^n$ and we want to recapture the value of f at a fixed point α (the amount of information does not increase). In this note we give explicit formulas that solve this recovery problem.

Let us agree that every point α, α_k, etc. will belong to U. Our main result is the following

THEOREM. *Let $\alpha_1, \ldots, \alpha_n$ be n distinct points from U. Set*

$$D = \left(\frac{1}{1 - \alpha_i \bar{\alpha}_j}\right)_{i,j=1}^n$$

and let D_k be the matrix obtained by exchanging every entry in the kth column of D by 1. If $c_k = \det D_k / \det D$ then for every $f \in H^1$ we have

$$\left| f(0) - \sum_1^n c_k f(\alpha_k) \right| \leqslant \|f\|_{H^1} \prod_1^n |\alpha_k|. \tag{1}$$

Of course, this implies that, for all of our spaces H and $f \in H$,

$$\left| f(0) - \sum_1^n c_k f(\alpha_k) \right| \leqslant \|f\|_H \prod_1^n |\alpha_k|,$$

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and the Blaschke product

$$h(z)=\prod_1^n \frac{z-\alpha_k}{1-\bar{\alpha}_k z}$$

shows that this is the best possible result; namely, for any constants d_k we have

$$\left|h(0)-\sum_1^n d_k h(\alpha_k)\right| = \|h\|_H \prod_1^n |\alpha_k| .$$

Our results has several consequences. First of all for the values of f at an arbitrary point α we can deduce, by an application of (1) to

$$f^*(z)=f\left(\frac{z+\alpha}{1+\bar{\alpha}z}\right) \quad \text{and} \quad \alpha_k^* = \frac{\alpha_k-\alpha}{1-\bar{\alpha}\alpha_k},$$

that with

$$D(\alpha)=\left(\frac{(1-\alpha\bar{\alpha}_j)(1-\bar{\alpha}\alpha_i)}{(1-|\alpha|^2)(1-\alpha_i\bar{\alpha}_j)}\right)^n_{i,j=1}$$

and

(2) $$c_k(\alpha)=\det(D(\alpha))_k/\det D(\alpha),$$

the inequality

(3) $$\left|f(\alpha)-\sum_1^n c_k(\alpha)f(\alpha_k)\right| \leqslant \|f\|_{H^1}\frac{1+|\alpha|}{1-|\alpha|}\prod_1^n\left|\frac{\alpha_k-\alpha}{1-\bar{\alpha}\alpha_k}\right| \qquad (f\in H^1)$$

holds. This implies that if $\{\alpha_{nk}\}$ is a point system such that $\alpha_{nk}\neq\alpha_{nj}$ if $k\neq j$ and $\Sigma_{k=1}^n(1-|\alpha_{nk}|)$ tends to infinity as $n\to\infty$, then to every $\alpha\in U$ there are constants c_{nk} such that for every $f\in H^1$,

(4) $$\lim_{n\to\infty}\sum_{k=1}^n c_{nk}f(\alpha_{nk})=f(\alpha).$$

More generally, we have

Corollary 1. *If $\alpha_{nk}\neq\alpha_{nj}$ for $k\neq j$ and $\Sigma_{k=1}^n(1-|\alpha_{nk}|)\to\infty$ as $n\to\infty$, then to every $L\in H^*$ there are constants $\{c_{nk}\}_{n=1,1\leqslant k\leqslant n}^\infty$ such that*

(5) $$\lim_{n\to\infty}\sum_{k=1}^n c_{nk}f(\alpha_{nk})=Lf$$

holds for every $f\in H$.

Thus, the value of f at any fixed point can be determined by the above formulas. However, we can say much more. Let

$$\omega(f,\delta)_H=\sup_{0\leqslant h\leqslant\delta}\|f(\circ)-f(\circ e^{ih})\|_H$$

be the H-modulus of continuity of f. We have

COROLLARY 2. *Let $\{\alpha_{nk}\}$ ($\alpha_{nk} \neq \alpha_{nj}$ for $k \neq j$) be a point system and let $s_n = \sum_{k=1}^n (1 - |\alpha_{nk}|)$. For each n there are polynomials p_{nk} of degree at most s_n such that for every $f \in H$,*

$$\left\| f - \sum_{k=1}^{n} f(\alpha_{nk}) p_{nk} \right\|_H \leqslant K_H \left(e^{-s_n/16} \| f \|_H + \omega\left(f, e^{-s_n/16} \right)_H \right). \tag{6}$$

Thus, if $s_n \to \infty$, then the operators

$$A_n f(z) = \sum_{k=1}^{n} f(\alpha_{nk}) p_{nk}(z) \tag{7}$$

approximate every $f \in H$ in H-norm.

This corollary is in sharp contrast with a result of G. Somorjai [3] asserting that there are no operators A_n of the form (7) with $|\alpha_{nk}| = 1$ such that $\| A_n f - f \|_{\sup} \to 0$ as $n \to \infty$ for every $f \in A$. We can see that if we move the nodes α_{nk} into U the situation changes radically.

We emphasize that both the constants c_{nk} in (4) and (5) and the polynomials p_{nk} in (6) can be effectively constructed (see the proofs below).

Finally, we mention that our results can be extended to the case when the nodes may coincide. Naturally, we have to then use the values of the higher derivatives of f as well. Let us consider, e.g., (1).

COROLLARY 3. *Let $\alpha_1, \ldots, \alpha_m$ be m distinct points and $n_1, \ldots, n_m$ nonnegative integers. Then there are constants $\{d_{kj}\}_{k=1,\, 0 \leqslant j \leqslant n_k}^{m}$ such that*

$$\left| f(0) - \sum_{k=1}^{m} \sum_{j=0}^{n_k} d_{kj} f^{(j)}(\alpha_k) \right| \leqslant \| f \|_{H^1} \prod_{1}^{m} |\alpha_k|^{n_k + 1} \qquad (f \in H^1).$$

Here

$$d_{kj} = \det D_{n_1 + \cdots + n_{k-1} + k + j} / \det D \qquad (1 \leqslant k \leqslant m, 0 \leqslant j \leqslant n_k),$$

where D is the square matrix of size $n_1 + \cdots + n_m + m$ with

$$\left(z^r / (1 - \alpha_k z)^{r+1} \right)^{(s)} \Big|_{z = \overline{\alpha_j}}$$

at the $(n_1 + \cdots + n_{k-1} + k + r,\ n_1 + \cdots + n_{j-1} + j + s)$ position $(1 \leqslant k, j \leqslant m, 0 \leqslant r \leqslant n_k, 0 \leqslant s \leqslant n_j)$.

It would be interesting to know if the assumption "$\sum_1^n (1 - |\alpha_{nk}|) \to \infty$ as $n \to \infty$" is necessary in Corollaries 1 and 2. Clearly, if $\alpha_{nk} = \alpha_k$ ($n \geqslant k$) then

$$\lim_{n \to \infty} \sum_{1}^{n} (1 - |\alpha_{nk}|) = \sum_{1}^{\infty} (1 - |\alpha_k|) = \infty$$

should be satisfied if we want to conclude convergence (at least for $H = H^p$, $1 \leqslant p < \infty$). In the general case we know only the trivial necessary condition $\sum_1^n (1 - |\alpha_{nk}|) \geqslant c > 0$ (if this is not satisfied then there is an $f \in H^1$, $f \neq 0$ such that $f(\alpha_{nk}) = 0$ $(1 \leqslant k \leqslant n)$ for infinitely many n).

PROOF. Let $e_k = e^{ikt}$ and $p_k(t) = (1 - \alpha_k e^{-it})^{-1}$. Clearly,

$$f(\alpha_k) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}) p_k(t)\, dt$$

for every $f \in H^1$. First we compute

$$d = \operatorname{dist}_{\sup}\big(\operatorname{Span}(\{p_k\}_1^n, \{e_k\}_1^\infty); 1\big).$$

By the Hahn-Banach theorem (see also [**2**, p. 71])

$$d = \sup_{\substack{L(e_k)=L(p_k)=0,\, \|L\|\leq 1\\ L\in C_{2\pi}^*}} |L1|.$$

Every $L \in C_{2\pi}^*$ is given by a complex Borel-measure μ:

$$Lg = \int_0^{2\pi} g(t)\, d\mu(t), \qquad \|L\| = \|\mu\| \quad (g \in C_{2\pi}).$$

Since for the L's in the supremum we have $L(e_k) = 0$ for $k = 1, 2, \ldots$ by the F. and M. Riesz theorem [**1**, p. 47] this yields that μ is absolutely continuous with respect to the Lebesgue-measure. It follows that for some $f \in H^1$ we have

$$Lg = \frac{1}{2\pi}\int_0^{2\pi} g(t) f(e^{it})\, dt \qquad (g \in C_{2\pi}),$$

$$\|L\| = \int_0^{2\pi} d|\mu| = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|\, dt = \|f\|_{H^1}$$

so

$$d = \sup_{f\in H^1,\, \|f\|_{H^1}\leq 1,\, f(\alpha_k)=0} |f(0)|.$$

Jensen's inequality [**1**, p. 51]

$$\log|f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{it})|\, dt \qquad (f \in H^1)$$

applied to the function

$$f(z)\prod_1^n \left(\frac{z-\alpha_k}{1-\bar{\alpha}_k z}\right)^{-1}$$

gives

$$\log|f(0)| \leq \sum_1^n \log|\alpha_k| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{it})|\, dt$$

so

$$\begin{aligned} d &\leq \sup_{f\in H^1,\, \|f\|_{H^1}\leq 1,\, f(\alpha_k)=0} \left(\prod_1^n |\alpha_k|\right) \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{it})|\, dt\right) \\ &\leq \sup_{f\in H^1,\, \|f\|_{H^1}\leq 1} \left(\prod_1^n |\alpha_k|\right) \|f\|_{H^1} = \prod_1^n |\alpha_k|. \end{aligned}$$

Now let us consider $L^2_{2\pi}$ with the inner product $(f, g) = (2\pi)^{-1}\int_0^{2\pi} f(t)\overline{g(t)}\,dt$. If

$$d^* = \operatorname{dist}_{L^2_{2\pi}}\left(\operatorname{Span}\left(\{\overline{p_k}\}_1^n, \{\overline{e_k}\}_1^\infty\right); 1\right)$$

then clearly $d^* \leqslant d$. The function

$$h^*(t) = \prod_1^n \frac{e^{it} - \alpha_k}{1 - \bar{\alpha}_k e^{it}}$$

is orthogonal to Span $(\{\overline{p_k}\}_1^n, \{\overline{e_k}\}_1^\infty)$ and has norm 1, so $d^* \geqslant |(h^*, 1)| = \prod_1^n |\alpha_k|$, i.e. we have $d = d^* = \prod_1^n |\alpha_k|$.

By what we have proven above to every $\varepsilon > 0$ there are a trigonometric polynomial $Q^\varepsilon \in \operatorname{Span}(\{e_k\}_1^\infty)$ and numbers $c_1^\varepsilon, \ldots, c_n^\varepsilon$ such that

$$d \leqslant \left\| 1 - \sum_{k=1}^n \overline{c_k^\varepsilon}\, \overline{p_k} - \overline{Q^\varepsilon} \right\|_{\sup} \leqslant d + \varepsilon$$

and this implies

$$(d \leqslant)\left\| 1 - \sum_{k=1}^n \overline{c_k^\varepsilon}\, \overline{p_k} - \overline{Q^\varepsilon} \right\|_{L^2_{2\pi}} < d + \varepsilon.$$

But then the conjugate of $\sum_1^n c_k^\varepsilon p_k + Q^\varepsilon$ converges as $\varepsilon \to 0$ to the orthogonal projection of 1 onto $\operatorname{Span}(\{\overline{p_k}\}_1^n, \{\overline{e_k}\}_1^\infty)$ and, hence, taking also into account that $(\overline{p_k}, \overline{p_j}) = (1 - \overline{\alpha_k}\alpha_j)^{-1}$, we get

$$1 - \sum_{k=1}^n \overline{c_k^\varepsilon}\left(1 - \overline{\alpha_k}\,\alpha_j\right)^{-1} = \left(1 - \sum_{k=1}^n \overline{c_k^\varepsilon}\, \overline{p_k} - \overline{Q^\varepsilon}, \overline{p_j}\right) \to 0 \qquad (\varepsilon \to 0).$$

This implies at once that every c_k^ε converges to a c_k as $\varepsilon \to 0$ and these c_k satisfy

$$\sum_{k=1}^n c_k\left(1 - \alpha_k \overline{\alpha_j}\right)^{-1} = 1, \qquad j = 1, 2, \ldots, n,$$

from which $c_k = \det D_k / \det D$. Note that $\det D \neq 0$ since D is the Gramm matrix of the linearly independent set $\{p_k\}_1^n$.

Now let $q = 1 - \sum_1^n c_k p_k$. If $\varepsilon > 0$ then there is a polynomial $Q_\varepsilon \in \operatorname{Span}(\{e_k\}_1^\infty)$ such that $|g(t) + Q_\varepsilon(t)| \leqslant d + \varepsilon$ for all t (see the consideration above). Hence for every $f \in H^1$ we have

$$\left| f(0) - \sum_1^n c_k f(\alpha_k) \right| = |(f \circ e_1, \bar{q})| = |(f \circ e_1, \overline{q + Q_\varepsilon})|$$

$$\leqslant \|f\|_{H^1} \|q + Q_\varepsilon\|_{\sup} \leqslant \|f\|_{H^1}(d + \varepsilon) = \|f\|_{H^1}\left(\prod_1^n |\alpha_k| + \varepsilon\right);$$

letting ε tend to zero we obtain (1). The proof is complete.

To prove Corollary 2 let $\alpha_1, \ldots, \alpha_n$ be n distinct points and $s = \sum_1^n (1 - |\alpha_k|)$. For any integer $r \geqslant 0$ set

$$c_{k,r} = \frac{1}{\pi i} \int_{|\alpha| = 1/2} c_k(\alpha) \alpha^{-r-1}\, d\alpha$$

where $c_k(\alpha)$ is the constant defined in (2). It is easy to see that

$$|(\alpha_k-\alpha)(1-\bar{\alpha}\alpha_k)^{-1}|\leqslant 1-\tfrac{1}{4}(1-|\alpha_k|)$$

whenever $|\alpha|\leqslant 1/2$; hence, by (3),

$$\left|f(\alpha)-\sum_1^n c_k(\alpha)f(\alpha_k)\right|\leqslant\frac{1+|\alpha|}{1-|\alpha|}\|f\|_{H'}\prod_1^n(1-(1-|\alpha_k|)/4)$$
$$\leqslant 4\|f\|_{H'}\exp(-s/4)$$

for every $|\alpha|=1/2$. Dividing both sides by $\pi i\alpha^{r+1}$ and integrating on $|\alpha|=1/2$, we obtain

$$\left|\frac{1}{r!}f^{(r)}(0)-\sum_{k=1}^n c_{k,r}f(\alpha_k)\right|\leqslant\|f\|_{H'}2^{r+3}\exp(-s/4). \tag{4}$$

Let

$$\sigma_m(f;z)=\sum_{r=0}^{m-1}\frac{f^{(r)}(0)}{r!}z^r+\sum_{r=m}^{2m}2\left(1-\frac{r}{2m}\right)\frac{f^{(r)}(0)}{r!}z^r$$

be the de la Vallée Poussin means of the Taylor expansion of f and let us consider the corresponding polynomials

$$p_k(z)=\sum_{r=0}^{m-1}c_{k,r}z^r+\sum_{r=m}^{2m}2\left(1-\frac{r}{2m}\right)c_{k,r}z^r \qquad (1\leqslant k\leqslant n).$$

By (4)

$$\left\|\sigma_m(f)-\sum_1^n f(\alpha_k)p_k\right\|_H\leqslant\|f\|_H\sum_{r=0}^{2m}2^{r+3}\exp(-s/4)$$
$$\leqslant 2^{2m+4}\|f\|_H\exp(-s/4).$$

Putting $m=[s/16]$, and taking into account that by the well-known properties of the de la Vallée Poussin kernels and Jackson's approximation theorem [**4**, pp. 524, 260–263]

$$\|\sigma_m(f)-f\|_H\leqslant K_H\omega\left(f,\frac{1}{m}\right)_H,$$

we obtain

$$\left\|f-\sum_1^m f(\alpha_k)p_k\right\|_H\leqslant K_H\left(e^{-s/16}\|f\|_H+\omega(f,e^{-s/16})_H\right),$$

which proves Corollary 2.

Corollary 1 is an immediate consequence of Corollary 2, since for $c_{nk}=Lp_{nk}$ we have

$$\left|Lf-\sum_{k=1}^n c_{nk}f(\alpha_k)\right|\leqslant K_H\|L\|\left(e^{-s/16}\|f\|_H+\omega(f,e^{-s/16})_H\right).$$

Corollary 3 could be deduced from the Theorem by a limiting argument, but it is easier to modify the proof of the latter so as to directly verify Corollary 3. Indeed, if $p_{k,j} = e^{-ijt}(1 - \alpha_k e^{-it})^{-j-1}$, then $(f \circ e_1, \overline{p_{k,j}}) = f^{(j)}(\alpha_k)$ $(f \in H^1)$ and exactly as in the proof of our Theorem we get

$$\operatorname{dist}_{\sup}\left(1 - \sum_{k=1}^{n} \sum_{j=0}^{n_k} c_{kj} p_{kj};\ \operatorname{Span}\left(\{e_k\}_1^\infty\right)\right) = \prod_{k=1}^{n} |\alpha_k|^{n_k+1},$$

and the proof can be completed as above. We omit the details.

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