PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 4, April 1984

THE SET OF ZEROES OF AN "ALMOST POLYNOMIAL" FUNCTION

Y. YOMDIN

ABSTRACT. Let f be a smooth function on the unit n-dimensional ball, with the C^0-norm, equal to one. We prove that if for some $k \geqslant 2$, the norm of the kth derivative of f is bounded by 2^{-k-1}, then the set of zeroes Y of f is similar to that of a polynomial of degree $k-1$. In particular, Y is contained in a countable union of smooth hypersurfaces; "many" straight lines cross Y in not more than $k-1$ points, and the $n-1$-volume of Y is bounded by a constant, depending only on n and k.

The result proved in this note gives an example of a rather general phenomenon: if a differentiable function is sufficiently close to a polynomial of degree $k-1$, i.e. if all the partial derivatives of order k are sufficiently small, then this function looks like a polynomial not only in C^0-norm, but also in many regards concerning its topological and singular structure. Many other examples are given in [**5**].

We consider infinitely smooth functions defined on bounded open domains in Euclidean spaces. If no additional assumptions are made, any closed set can be the set of zeroes of some such function.

Let R^s, $s = 1, \ldots,$ denote the s-dimensional Euclidean space with the standard coordinates and the standard scalar product. Let B_r^s denote the open ball of radius r, centered at the origin of R^s. We consider all the spaces of linear and multilinear mappings of Euclidean spaces equipped with the corresponding Euclidean norms.

For an open domain $D \subset R^n$ and $f \in C^\infty(D)$, let $M_k(f, D)$ denote the $\sup_{y \in D} \|d^k f(y)\|$, $k = 0, 1, \ldots,$ $d^k f$—the kth derivative of f.

DEFINITION 1. A bounded open domain $D \subset R^n$ is said to be admissible, if there exist constants N_k, $k = 0, 1, \ldots,$ with the following property: for any $k \geqslant 0$ and for any $f \in C^\infty(D)$ there is a sequence $h_1^k, \ldots, h_i^k, \ldots \in C^\infty(R^n)$, such that $M_k(h_i^k, R^n) \leqslant N_k \cdot M_k(f, D)$, $i = 1, \ldots,$ $k = 0, 1, \ldots,$ and the sets $V_i = \{y \in D, f(y) = h_i^k(y)\}$ increase and cover D.

The infimum of N_k above is denoted by $N_k(D)$.

DEFINITION 2. Let $D \subset R^n$ be an admissible domain. Let $r(D)$ be the radius of the minimal ball in R^n containing D. Define the numbers $\alpha_k(D)$, $k = 0, 1, \ldots,$ by $\alpha_k(D) = 1/(r(D))^k \cdot N_k(D) \cdot 2^{k+1}$.

We do not study here the precise geometric conditions for a given domain to be admissible. Using the Whitney extension theorem [**4**], one can easily show, for example, that any domain with a piecewise-smooth boundary is admissible.

Received by the editors May 3, 1983.
1980 *Mathematics Subject Classification*. Primary 41A10, 26D10.

(In fact, using [3], (3.2.1) and Remark 3.4, one can show that any domain D, satisfying the following condition, is admissible:

$$D = \bigcup_{i=1}^{\infty} D_i, \quad \text{where } D_i = \overline{D}_i, D_1 \subset D_2 \subset \cdots ,$$

and for some $K > 0$, any two points x, y in D_i, $i = 1, 2, \ldots$, can be joined in D_i by some curve of the length $\leqslant K\|x - y\|$.)

THEOREM 3. *Let $D \subset R^n$ be an admissible domain, and let $f \in C^\infty(D)$. If for some $k \geqslant 2$, $M_k(f, D) < \alpha_k(D) \cdot M_0(f, D)$, then*

(i) *$Y(f) = f^{-1}(0)$ is contained in a countable union of compact smooth hypersurfaces in D.*

(ii) *There is an open subset U in the set of all the straight lines in R^n, such that the lines of U cover all the space R^n and any such line intersects $Y(f)$ in not more than $k - 1$ points.*

(iii) *The $n - 1$-dimensional Hausdorff measure $m_{n-1}(Y(f))$ satisfies $m_{n-1}(Y(f)) \leqslant C(n, k) \cdot (r(D))^{n-1}$, where the constants $C(n, k)$ depend only on n and k.*

PROOF. If we can cover $Y(f)$ by an increasing sequence of subsets Y_i satisfying (i) –(iii) (for which U in (ii) remains the same), then $Y(f)$ itself has required properties. But according to Definition 1, we can take $Y_i = Y(f) \cap Y(h_i^k)$, if the theorem is valid for functions h_i^k. Thus it is sufficient to prove Theorem 3 only for $D = B_r^n$, with $\alpha_k(B_r^n) = 1/2^{k+1} \cdot r^k$.

So let $f \in C^\infty(B_r^n)$ and let $M_k(f, B_r^n) < (1/2^{k+1} \cdot r^k) \cdot M_0(f, B_r^n)$. Below we denote $M_i(f, B_r^n)$ shortly by M_i. Replacing, if necessary, f by $-f$, we can assume that $\sup_{x \in B_r^n} f(x) = M_0$. Let

$$V = \left\{ x \in B_r^n, f(x) > \tfrac{1}{2} M_0 \right\}.$$

LEMMA 4. *V contains some ball B of radius Kr, where $K = 1/20(k - 1)^2$.*

PROOF. First we note that $r \cdot M_1 \leqslant 4(k - 1)^2 \cdot M_0$. Indeed, let P be the Taylor polynomial of degree $k - 1$ of f at some $x_0 \in B_r^n$. For any $x \in B_r^n$,

$$|f(x) - P(x)| \leqslant (1/k!) \cdot r^k \cdot M_k < \left(1/2^{k+1}k!\right) \cdot M_0,$$

$$\|df(x) - dP(x)\| \leqslant (1/(k - 1)!) \cdot r^{k-1} \cdot M_k < \left(1/2^{k+1}(k - 1)! \cdot r\right) \cdot M_0.$$

Now, by the Markov inequality (see e.g. [2, Theorem VI]) for a polynomial P of degree $k - 1$, $r \cdot M_1(P) \leqslant 2(k - 1)^2 \cdot M_0(P)$. Hence

$$\begin{aligned} r \cdot M_1 &\leqslant r \cdot M_1(P) + \left(1/2^{k+1}(k - 1)!\right) M_0 \\ &\leqslant 2(k - 1)^2 \cdot M_0(P) + \left(1/2^{k+1}(k - 1)!\right) M_0 \\ &\leqslant 2(k - 1)^2\left(1 + \left(1/2^{k+1}k!\right)\right) M_0 + \left(1/2^{k+1}(k - 1)!\right) M_0 \\ &< 4(k - 1)^2 M_0. \end{aligned}$$

Now take some $x_1 \in V$, such that $f(x_1) > 0.9M_0$. By the inequality above, any point $y \in B_r^n$, such that $\|y - x_1\| < 0.1r/(k - 1)^2$, also belongs to V, i.e. V contains the intersection of B_r^n with the ball of radius $0.1r/(k - 1)^2$, centered at x_1, which in turn contains some ball of radius $0.05r/(k - 1)^2$.

Let U be the set of all the straight lines in R^n passing through the ball B. Clearly, U is an open set, whose lines cover R^n. Notice also, that in proof of a general version of Theorem 3, we can assume that already for h_1^k there is a point in D, where $h_1^k = f > 0.9M_0$. Taking this point as x_1, we obtain B and U not depending on h_i^k.

To prove that any line of U intersects $Y(f)$ in not more than $k - 1$ points, we need the following elementary inequality (which is the one-dimensional case of Theorem 3):

LEMMA 5. *Let $\phi \in C^\infty((a, b))$, $k \geqslant 2$. If $M_k(\phi) < M_0(\phi)/(b - a)^k$, then ϕ has in (a, b) not more than $k - 1$ zeroes (counted with their multiplicities).*

PROOF. Let us show that if ϕ has in (a, b) more than $k - 1$ zeroes, then $M_0(\phi) \leqslant (b - a)^k M_k(\phi)$. For $k = 1$ this is the mean value theorem. For $k \geqslant 2$ apply the lemma to ϕ'.

Now let $L \in U$ and let $\phi = f/L \cap B_r^n$. By construction of U, $M_0(\phi) \geqslant 0.5M_0$. On the other hand, $M_k(\phi) \leqslant M_k$, and we have $M_k(\phi) \leqslant M_k < (1/2(2r)^k) \cdot M_0 \leqslant M_0(\phi)/(2r)^k$. Since the length of the interval $L \cap B_r^n$ is not greater than $2r$, ϕ satisfies conditions of Lemma 5, and hence the number of zeroes of ϕ is not greater than $k - 1$. This proves part (ii) of Theorem 3.

To prove (i), consider some $x_0 \in Y(f)$. Let $L \in U$ pass through x_0. By (ii), the multiplicity q of zero of f/L at x_0, does not exceed $k - 1$. Choose a coordinate system $y_1, \ldots, y_n$ at x_0, such that L coincides with the last axis. According to the Malgrange-Weierstrass preparation theorem [3], there exists a neighborhood W of x_0 in which f can be written as $f = gh$, with $g(x_0) \neq 0$, and

$$h(y_1, \ldots, y_n) = y_n^q + \beta_{q-1}(y_1, \ldots, y_{n-1}) y_n^{q-1} + \cdots + \beta_0(y_1, \ldots, y_{n-1}),$$

where $\beta_i \in C^\infty(Z)$, Z being an appropriate neighborhood of x_0 in the hyperplane $y_n = 0$.

Hence $Y(f) \cap W$ coincides with $Y(h) = \{h = 0\}$.

LEMMA 6. *Let $u_a(y) = y^q + a_{q-1}y^{q-1} + \cdots + a_0$ be a polynomial with real coefficients $a = (a_0, \ldots, a_{q-1}) \in R^q$. The space R^q can be represented as a countable union of closed (defined only by equalities and nonstrict inequalities) semialgebraic subsets A_i with the following property: for any i there are smooth functions $\psi_1^i, \ldots, \psi_{k_i}^i \colon R^q \to R$ such that for any $a \in A_i$ the polynomial u_a has exactly k_i distinct real roots $z_1(a) < \cdots < z_{k_i}(a)$ and $z_j(a) = \psi_j^i(a), j = 1, \ldots, k_i$.*

PROOF. For any s-tuple of integers $(q_1, \ldots, q_s)$ such that $q_1 + \cdots + q_s \leqslant q$, let $A^{q_1, \ldots, q_s} \subset R^q$ be the set of all $a \in R^q$, for which u_a has exactly s distinct real roots $z_1(a) < \cdots < z_s(a)$ of multiplicities $q_1, \ldots, q_s$ respectively. $A^{q_1, \ldots, q_s}$ are semialgebraic subsets of R^q (see e.g. [1]).

Now at each $a \in A^{q_1, \ldots, q_s}$ the root $z_j(a)$ of u_a coincides with the simple root $w_j(a)$ of $d^{q_j - 1}u_a/dy^{q_j - 1}$; but $w_j(a)$ is a smooth (in fact, analytic) function, defined in some open neighborhood of $A^{q_1, \ldots, q_s}$. By the Whitney theorem [4] for any closed subset F in $A^{q_1, \ldots, q_s}$ the restriction $z_j/F = w_j/F$ is exendable to a smooth function $\bar{w}_j$ defined on R^q.

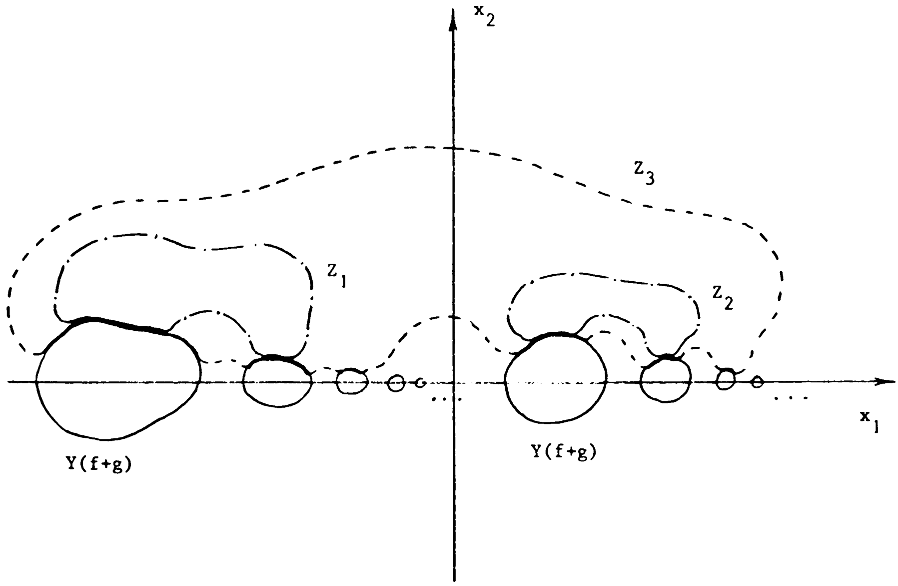

FIGURE 1

Now replacing each inequality of the form $p(a) > 0$ in the definition of $A^{q_1,\ldots,q_s}$ as a semialgebraic set by inequalities $p(a) \geqslant 1/m$, $m = 1, 2, \ldots,$ we represent $A^{q_1,\ldots,q_s}$ as a countable union of closed semialgebraic sets, for each of them functions ψ_j^i being defined by extensions $\overline{w}_j$ as above. Lemma 6 is proved.

Let us consider the mapping $\beta = (\beta_0, \ldots, \beta_{q-1})\colon Z \to R^q$, where β_i are the coefficients of the polynomial h, defined above. Let $D_i = \beta^{-1}(A_i)$. The closed subsets D_i cover Z and by Lemma 6, $Y(f) \cap W = Y(h)$ is contained in the union of all the hypersurfaces Z_j^i, defined by the equations $y_n = \psi_j^i \circ \beta(y_1, \ldots, y_{n-1})$. Compactifying Z_j^i, we obtain the required compact hypersurfaces, whose union contains $Y(f) \cap W$, and covering B_r^n by a contable number of neighborhoods W, we prove the part (i) of the theorem.

Part (iii) follows from the following easy integral-geometric inequality:

LEMMA 7. *Let Y be a smooth hypersurface in B_r^n. If any straight line passing through a fixed ball B of radius r' in B_r^n intersects Y in at most p points, then*

$$m_{n-1}(Y) \leqslant Q(n, r'/r) \cdot p \cdot r^{n-1},$$

where the constant Q depends only on n and on the ratio r'/r of radii of the balls B and B_r^n.

By standard measure-theoretic arguments, this lemma can be applied also to the set $Y(f)$, containing in a countable union of smooth hypersurfaces. Since the radius of the ball B constructed in Lemma 4 is $K \cdot r$, with K depending only on k, inequality (iii) follows.

REMARK 1. We can include in Theorem 3 also the case $k = 1$. In this case conditions of the theorem imply that $Y(f)$ is empty.

REMARK 2. The assumption of an infinite differentiability of functions is not essential. We can assume f in Theorem 3 to be differentiable only $k+1$ times.

EXAMPLE. Consider $f(x_1, x_2) = x_2^2$. Hence $d^3 f \equiv 0$. Adding to f an arbitrarily small smooth function $g(x_1)$ we can obtain any set of zeroes $Y(f+g)$ of the type, shown on Figure 1, where also compact hypersurfaces Z are shown.

The author would like to thank SFB-40, Universität Bonn, for its kind hospitality.

REFERENCES

1. M. Coste, *Ensembles semi-algebriques*, Geometrie Algebrique Reelle et Formes Quadratiques, Lecture Notes in Math., Springer, Berlin and New York, vol. 952, 1982, pp. 109–138.

2. O. D. Kellogg, *On bounded polynomials in several variables*, Math. Z. **27** (1928), 55–64.

3. B. Malgrange, *Ideals of differentiable functions*, Oxford Univ. Press, London, 1966.

4. H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), 63–89.

5. Y. Yomdin, *The geometry of critical and near-critical values of differentiable mappings*, Math. Ann. **264** (1983), 495–515.

MAX - PLANCK INSTITUT FÜR MATHEMATIK, GOTTFRIED - CLAREN STRASSE 26, 5300 BONN 3, FEDERAL REPUBLIC OF GERMANY (Current address)

DEPARTMENT OF MATHEMATICS, BEN - GURION UNIVERSITY OF THE NEGEV, BEER - SHEVA 84120, ISRAEL