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MACKEY FUNCTORS AND G-COHOMOLOGY

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ABSTRACT. The Bredon-Illman cohomology of universal G-spaces associated with a family of subgroups is related to derived functors in three fundamental categories of G-orbits. Analogous results for the G-cohomology of classifying spaces for G-covering spaces are also obtained.

Introduction. The purpose of this note is to complete the work in [**W1** and **W2**], and to provide the necessary basic results required for the computations in ordinary RO(G)-graded equivariant cohomology to appear in [**LMMW**].

We consider here three categories of G-orbits fundamental to equivariant cohomology theory, and the relationship of associated derived functors to the Bredon-Illman cohomology of universal G-spaces. This results in a generalization of the work in [**W1** and **W2**], and also in some algebraic information (Corollary 4.3).

Our method for constructing chain complexes was inspired in part by J. P. May, and readily generalizes to give explicit chain complexes for ordinary RO(G)-graded cohomology. Details will appear in [**LMMW**].

The author is grateful to Leonard Scott for many stimulating conversations on the subject.

1. Three topological categories of G-orbits. First we need some notation. Fix a compact Lie group G, and an orthogonal G-module U such that:

(i) Each finite-dimensional orthogonal G-module occurs, up to isomorphism, infinitely often as an invariant subspace of U.

(ii) U is the union of its finite-dimensional G-invariant submodules.

The notation $V < U$ will be used to indicate that V is a finite-dimensional G-invariant submodule of U, while $H \subset G$ will always refer to a closed subgroup of G. The one-point compactification of $V < U$ will be denoted by S^V, while S^n will be understood to have the trivial G-action.

The category of compactly generated weak Hausdorff based G-spaces will be denoted by $G\mathcal{T}$, and its associated homotopy category by $hG\mathcal{T}$, two G-maps being equivalent if they are homotopic through G-maps. If $G = 1$, denote $G\mathcal{T}$ by $\mathcal{T}$.

If X and Y are in $G\mathcal{T}$, then $h\mathcal{T}(X,Y)$ is acted upon by G via conjugation of representing classes. Denote by $h\mathcal{T}^G$ the category whose objects are those of $G\mathcal{T}$ and whose morphisms are given by

$$h\mathcal{T}^G(X,Y) = [X,Y]^G,$$

where $[\ ,\ ]$ denotes $h\mathcal{T}(\ ,\ )$, for any pair of objects (X,Y) in $G\mathcal{T}$, and where the superscript G denotes the G-fixed set. Thus a morphism is a homotopy class of maps $f\colon X \to Y$ such that gfg^{-1} is homotopic to f for each $g \in G$. $[\ ,\ ]_G$, on the other hand, will denote the set of G-homotopy classes of G-maps as is customary.

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Let $\mathcal{F}$ be a family of closed subgroups $H \subset G$, and define categories $\mathcal{B}(\mathcal{F})$, $\mathcal{M}(\mathcal{F})$ and $\mathcal{H}(\mathcal{F})$ as follows. The objects of each of these categories are the left coset spaces G/H with $H \in \mathcal{F}$, while the morphisms are given by

$$\begin{aligned}
\mathcal{B}(\mathcal{F})(G/H, G/K) &= \lim_{n}[S^n \wedge G/H_+, S^n \wedge G/K_+]_G,\\
\mathcal{M}(\mathcal{F})(G/H, G/K) &= \lim_{V<U}[S^V \wedge G/H_+, S^V \wedge G/K_+]_G,\\
\mathcal{H}(\mathcal{F})(G/H, G/K) &= \lim_{V<U}[S^V \wedge G/H_+, S^V \wedge G/K_+]^G.
\end{aligned}$$

Here the subscript $+$ denotes addition of a disjoint basepoint, the letters $\mathcal{B}$, $\mathcal{M}$, and $\mathcal{H}$ refer to Bredon, Mackey and Hecke, respectively (for reasons to be clarified below), and all limits in sight are colimits taken with respect to suspension. Note that

$$\mathcal{H}(\mathcal{F})(G/H, G/K) \cong \lim_{n}[S^n \wedge G/H_+, S^n \wedge G/K_+]^G,$$

since one may associate with any map $f\colon S^V \wedge G/H_+ \to S^V \wedge G/K_+$ the composite

$$S^n \wedge G/H_+ \to S^V \wedge G/H_+ \xrightarrow{f} S^V \wedge G/K_+ \to S^n \wedge G/K_+,$$

where $n = \dim V$, via some map $S^n \to S^V$ of degree one; this composite being in $\mathcal{H}(\mathcal{F})$ if the map f is.

There is an evident diagram of additive functors

$$\begin{tikzcd}
 & \mathcal{M}(\mathcal{F}) \arrow[dr, "\mathcal{H}\mathcal{M}"] & \\
\mathcal{B}(\mathcal{F}) \arrow[ur, "\mathcal{M}\mathcal{B}"] \arrow[rr, "\mathcal{H}\mathcal{B}"'] & & \mathcal{H}(\mathcal{F})
\end{tikzcd}$$

which one uses to pass from one category to another. When $\mathcal{F}$ contains a subgroup from each conjugacy class, we shall supress $\mathcal{F}$ from the notation. (Although this is not necessary, one may insist that, in general, $\mathcal{F}$ contain exactly one subgroup from each conjugacy class of closed subgroups, so that $\mathcal{B}$, $\mathcal{M}$, and $\mathcal{H}$ are unambiguous.) Denote by $\mathcal{A}b$ the category of abelian groups.

DEFINITION 1.1. Let $\mathcal{C} = \mathcal{B}, \mathcal{M}$ or $\mathcal{H}$. A $\mathcal{C}(\mathcal{F})$-functor is an additive contravariant functor $T\colon \mathcal{C}(\mathcal{F}) \to \mathcal{A}b$. When $\mathcal{C} = \mathcal{B}$ we refer to an $\mathcal{F}$-Bredon functor, and similarly for $\mathcal{M}$ and $\mathcal{H}$.

The collection of $\mathcal{C}(\mathcal{F})$-functors forms a category $F(\mathcal{C}(\mathcal{F}))$ whose morphisms are the natural transformations. Note that the functors above induce a commutative diagram:

$$\begin{tikzcd}
 & F(\mathcal{M}(\mathcal{F})) \arrow[dl, "\mathcal{M}\mathcal{B}^*"'] & \\
F(\mathcal{B}(\mathcal{F})) & & F(\mathcal{H}(\mathcal{F})) \arrow[ul, "\mathcal{H}\mathcal{M}^*"'] \arrow[ll, "\mathcal{M}\mathcal{B}^*"]
\end{tikzcd}$$

REMARKS 1.2. This notion of a Mackey functor is due to Lewis, May and McClure while our notion of a Bredon functor coincides with that of a (contravariant) coefficient system, due to Bredon and Illman (see [**B1** and **I1**]). When G is finite, a Hecke functor will be seen to be equivalent to a well-known algebraic functor—what Green refers to as a cohomological functor in [**G1**].

The objects of the categories $F(\mathcal{C}(\mathcal{F}))$ will become the coefficient systems of associated equivariant cohomology theories.

EXAMPLES 1.3. (i) With $\mathcal{C}$ as above, define

$$\mathcal{C}(\mathcal{F})_H : \mathcal{C}(\mathcal{F}) \to \mathcal{A}b$$

by $\mathcal{C}(\mathcal{F})_H(G/K) = \mathcal{C}(\mathcal{F})(G/K, G/H)$, with the action on morphisms taken to be the evident left one. The functors $\mathcal{C}(\mathcal{F})_H$ will be seen to be canonically projective, and will be the building blocks of our algebraic resolutions.

(ii) If X is a G-CW complex (in the sense of Matumoto [**M1**]), then one may define an associate d.g. $\mathcal{F}$-Bredon functor by

$$C_n(X)(G/H) = \lim_r [S^{n+r} \wedge G/H_+, \Sigma^r X^n/X^{n-1}]_G,$$

where X^n is the n-skeleton of X. One may then verify that $F(\mathcal{B}(\mathcal{F}))(C_*(X), T)$ is isomorphic with the Bredon-Illman cochain complex with coefficients in $T \in \mathcal{B}(\mathcal{F})$ (at least when $\mathcal{F}$ is closed under passage to subgroups under conjugacy). Similarly, one may define a d.g.-Mackey functor, $D_*(X)$, by

$$D_n(X)(G/H) = \lim_V [S^{n+V} \wedge G/H_+, \Sigma^V X^n/X^{n-1}]_G,$$

and verify that, for $T \in \mathcal{M}(\mathcal{F})$, $F(\mathcal{M}(\mathcal{F}))(D_*(X), T)$ once again gives the Bredon-Illman cochain complex for suitable $\mathcal{F}$. This approach is used by Lewis, May and McClure to set up ordinary RO(G)-graded cohomology. Note that the homotopy sequence for the pair (X^n, X^{n-1}) gives morphisms ∂_*: $C_n(X) \to C_{n-1}(X)$ and similarly for the $D_*(X)$.

(iii) If G if finite, then $\mathcal{B}(\mathcal{F})(G/H, G/K)$ is the free abelian group on $(G/K)^H \cong N_K(H)/K$, where $N_K(H) = \{g \in G : gHg^{-1} \subset K\}$. Further, $\mathcal{M}(\mathcal{F})(G/H, G/K)$ may be seen to be equivalent to $A(G/H \times G/K)$, where A denotes the Burnside functor of tom Dieck [**D1**]. (Also see [**L1**] for an equivalent formulation of $\mathcal{M}(\mathcal{F})$ in the finite case.) Finally,

$$\mathcal{H}(\mathcal{F})(G/H, G/K) = \mathrm{Hom}_{\mathbf{Z}G}(\mathbf{Z}G/H, \mathbf{Z}G/K) = (\mathbf{Z}G/K)^H,$$

where $\mathbf{Z}G/J$ denotes the free abelian group on the G-set G/J with the evident $\mathbf{Z}G$-action. This explains the choice of the names Bredon, Mackey and Hecke (also see [**RS**]).

(iv) When G is finite, one has further canonical examples of $\mathcal{C}(\mathcal{F})$-functors. If T is any coefficient system in the sense of Bredon, then T is clearly an $\mathcal{F}$-Bredon functor; any Mackey functor in the sense of tom Dieck is an $\mathcal{F}$-Mackey functor (by the work in [**L1**]), while any $\mathbf{Z}G$-module M determines an $\mathcal{F}$-Hecke functor $\hat{M}$ via the assignment $G/H \mapsto M^H \cong \mathrm{Hom}_{\mathbf{Z}G}(\mathbf{Z}G/H, M)$. Such functors are discussed in [**W1** and **W2**].

2. Homological algebra of $\mathcal{C}(\mathcal{F})$-functors. Here we construct explicit projective resolutions in $F(\mathcal{C}(\mathcal{F}))$, and give examples arising from equivariant topology, generalizing work in [**W1** and **W2**].

Recall the $\mathcal{C}(\mathcal{F})$-functors $\mathcal{C}(\mathcal{F})_H$ of Example 1.3.

LEMMA 2.1. *Let $\mathcal{C} = \mathcal{B}(\mathcal{F}), \mathcal{M}(\mathcal{F})$ or $\mathcal{H}(\mathcal{F})$, and let $H \in \mathcal{F}$. Then $\mathcal{C}(\mathcal{F})_H$ is a projective $\mathcal{C}(\mathcal{F})$-functor.*

PROOF. This is essentially an elaboration of [**W1**, 2.7], in which the result is proved for $\mathcal{C} = \mathcal{H}$. One first observes that, for each $T \in F(\mathcal{C}(\mathcal{F}))$ one has $F(\mathcal{C}(\mathcal{F}))(\mathcal{C}(\mathcal{F})_H, T) \cong T(G/H)$. To see this, define

$$\psi\colon F(\mathcal{C}(\mathcal{F}))(\mathcal{C}(\mathcal{F})_H, T) \to T(G/H)$$

by taking f to $f(G/H)(1_H)$, where 1_H is the identity morphism on G/H in $\mathcal{B}(\mathcal{F})$. Also define

$$\phi\colon T(G/H) \to F(\mathcal{B}(\mathcal{F}))(\mathcal{C}(\mathcal{F})_H, T)$$

by defining $\phi(t)(G/K)$ to be the composite

$$\mathcal{C}(\mathcal{F})_H(G/K) = \mathcal{C}(\mathcal{F})(G/K, G/H) \xrightarrow{\epsilon} T(G/K),$$

where $\epsilon(r) = T(r)(t)$ for $t \in T(G/H)$. Thus

$$\phi(t)(G/K)(x) = T(x)(t) \quad \text{for } x \in \mathcal{C}(\mathcal{F})_H(G/K).$$

That $\phi(t)$ is indeed a morphism of $\mathcal{C}(\mathcal{F})$-functors may be checked by a diagram chase, and that $\psi\phi = 1$ is clear. One checks that $\phi\psi = 1$ as follows. Given any morphism $f\colon \mathcal{C}(\mathcal{F})_H \to T$ and $x \in \mathcal{C}(\mathcal{F})_H(G/K)$ with $K \in \mathcal{F}$, the diagram

$$\begin{array}{ccc} \mathcal{C}(\mathcal{F})_H(G/H) & \xrightarrow{f(G/H)} & T(G/H) \\ {\scriptstyle \mathcal{C}(\mathcal{F})_H(x)}\downarrow & & {\scriptstyle T(x)}\downarrow \\ \mathcal{C}(\mathcal{F})_H(G/K) & \to & T(G/K) \end{array}$$

must commute, where $x \in \mathcal{C}(\mathcal{F})(G/K, G/H)$. Chasing 1_H around the diagram gives

$$f(G/K)(\mathcal{C}(\mathcal{F})_H(x))(1_H) = f(G/K)(x) = T(x)(f(G/H)(1_H)).$$

But this last expression is $\phi\psi(f)(G/K)(x)$, whence $\phi\psi = 1$ as claimed.

Projectivity of $\mathcal{C}(\mathcal{F})_H$ now follows formally; one observes that any diagram of the form

$$\begin{array}{ccc} & & T' \\ & \overset{g}{\nearrow} & \downarrow p \\ \mathcal{C}(\mathcal{F})_H & \xrightarrow{f} & T \end{array}$$

in $F(\mathcal{C}(\mathcal{F}))$ may be completed by setting $g = \phi(t)$ for any $t \in T'(G/H)$ such that $p(t) = f(G/H)(1_H)$. $\square$

Construction 2.2. Let $T \in F(\mathcal{C}(\mathcal{F}))$, and define $B_*(\mathcal{C}(\mathcal{F}), \mathcal{C}(\mathcal{F}), T)$ a d.g. $\mathcal{C}(\mathcal{F})$-functor by taking

$$\begin{aligned} B_n(\mathcal{C}(\mathcal{F}), \mathcal{C}(\mathcal{F}), T)(G/H) &= \Sigma \mathcal{C}(\mathcal{F})(G/H, G/H_0) \\ &\quad \otimes [\mathcal{C}(\mathcal{F})(G/H_0, G/H_1) \otimes \cdots \otimes \mathcal{C}(\mathcal{F})(G/H_{n-1}, G/H_n)] \otimes T(G/H_n), \end{aligned}$$

where the sum is taken over all distinct sequences $(H_0, \ldots, H_n)$ of subgroups in $\mathcal{F}$.

$$d_n\colon B_n(\mathcal{C}(\mathcal{F}), \mathcal{C}(\mathcal{F}), T) \to B_{n-1}(\mathcal{C}(\mathcal{F}), \mathcal{C}(\mathcal{F}), T)$$

is given by $d_n(x) = \sum_{i=0}^{n} (-1)^i F_i(x)$, where F_i is specified on generators by the formula

$$F_i(f[f_1, \ldots, f_n]t) = \begin{cases} f_1 \circ f[f_2, \ldots, f_n]t & \text{if } i = 0; \\ f[f_1, \ldots, f_{i+1} \circ f_i, \ldots, f_n]t & \text{if } 0 < i < n; \\ f[f_1, \ldots, f_{n-1}]T(f_n)(t) & \text{if } i = n. \end{cases}$$

Also define $\epsilon\colon B_0(\mathcal{C}(\mathcal{F}), \mathcal{C}(\mathcal{F}), T) \to T$ in $F(\mathcal{C}(\mathcal{F}))$ by setting $\epsilon(f[\ ]t) = T(f)(t)$.

PROPOSITION 2.3. *Let $T \in F(\mathcal{C}(\mathcal{F}))$ be such that each $T(G/H)$ is projective. Then $B_*(\mathcal{C}(\mathcal{F}), \mathcal{C}(\mathcal{F}), T)$ is a projective resolution of T in $F(\mathcal{C}(\mathcal{F}))$.*

PROOF. A contracting chain homotopy for each $B_*(\mathcal{C}(\mathcal{F}), \mathcal{C}(\mathcal{F}), T)(G/H)$ is given by

$$s_n(f[f_1, \ldots, f_n]t) = 1_H[f, f_1, \ldots, f_n]t.$$

To see that each $B_n(\mathcal{C}(\mathcal{F}), \mathcal{C}(\mathcal{F}), T)$ is projective, one observes that $\mathcal{C}(\mathcal{F})(G/J, G/K)$ is a free $\mathbf{Z}$-module for any J and K in $\mathcal{F}$, so that $B_n(\mathcal{C}(\mathcal{F}), \mathcal{C}(\mathcal{F}), T)$ is a sum of $\mathcal{C}(\mathcal{F})$-functors of the form $P \otimes F$ where F is free, $P = \mathcal{C}(\mathcal{F})_H$ for some $H \in \mathcal{F}$, and $(P \otimes F)(G/K) = P(G/K) \otimes F$ for $K \in \mathcal{F}$. □

DEFINITION 2.4. If T and T' are in $F(\mathcal{C}(\mathcal{F}))$, we define $H^*(T, T')_{\mathcal{C}(\mathcal{F})}$ to be the derived functor $\mathrm{Ext}^*_{\mathcal{C}(\mathcal{F})}(T, T')$ which, by Proposition 2.3, is the cohomology of $F(\mathcal{C}(\mathcal{F}))(B_*(\mathcal{C}(\mathcal{F}), \mathcal{C}(\mathcal{F}), T), T')$.

As a special case, one may take $\mathcal{C} = \mathcal{H}$, $\mathcal{F} = \{e\}$, $T = \hat{\mathbf{Z}}$ and $T' = \hat{R}$ for a $\mathbf{Z}G$-module R (as in Example 1.3(iv)). One then retrieves the cohomology of G with coefficients in R, since

$$F(\mathcal{H}(\mathcal{F}))(B_*, \hat{R}) = \mathrm{Hom}_{\mathbf{Z}G}(B_*(G/e), R),$$

where $B_* = B_*(\mathcal{H}(\mathcal{F}), \mathcal{H}(\mathcal{F}), \mathbf{Z})$ and $B_*(G/e)$ is a free resolution of $\mathbf{Z}$ (via the $\mathbf{Z}G$-action $g \cdot f[-]n = \hat{g} \circ f[-]n$, $\hat{g}\colon \mathbf{Z}G \hookleftarrow$ being multiplication of cosets $g'H \mapsto gg'H$).

REMARK 2.5. If X is any G-CW complex with cells of type $G/H \times D^m$ with $H \in \mathcal{F}$ and skeleta X^m, then the $\mathcal{C}(\mathcal{F})$-functors $D_*(X)$ and $C_*(X)$ are projective in $F(\mathcal{C}(\mathcal{F}))$, since they are sums of canonical projectives of the type just discussed.

3. Associated ordinary cohomology theories. Here, we construct cellular equivariant theories based on the categories $\mathcal{B}(\mathcal{F})$ and $\mathcal{M}(\mathcal{F})$, and comment on problems with $\mathcal{H}(\mathcal{F})$.

The following categorical constructions have been explored by Lewis, May and McClure in [**LMM**], and by the author in [**W4**].

If $\overline{M}$ and $\overline{N}$ are contravariant additive functors $\mathcal{C}(\mathcal{F}) \to \mathcal{A}b$ and if $\underline{N}\colon \mathcal{C}(\mathcal{F}) \to \mathcal{A}b$ is covariant, one may form

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}(\mathcal{F})}(\overline{M}, \overline{N}) &= \{\text{natural transformations } \overline{M} \to \overline{N}\}\\ &= F(\mathcal{C}(\mathcal{F}))(\overline{M}, \overline{N})\end{aligned}$$

and

$$\overline{M} \otimes_{\mathcal{C}(\mathcal{F})} \underline{N} = \left[\sum_{H \in \mathcal{F}} \overline{M}(G/H) \otimes \underline{N}(G/H)\right] / \sim,$$

where we define $m \otimes f_*(n) \sim mf^* \otimes n$ for f a morphism in $\mathcal{C}(\mathcal{F})$.

Referring to Example 1.3(ii), one may form the complexes

$$C^*(X, \overline{N})_{\mathcal{B}(\mathcal{F})} = \mathrm{Hom}_{\mathcal{B}(\mathcal{F})}(C_*(X), \overline{N})$$

and

$$C^*(X, \overline{N})_{\mathcal{M}(\mathcal{F})} = \mathrm{Hom}_{\mathcal{M}(\mathcal{F})}(D_*(X), \overline{N})$$

as well as their homology counterparts,

$$C_*(X, \underline{N})_{\mathcal{B}(\mathcal{F})} = C_*(X) \otimes_{\mathcal{B}(\mathcal{F})} \underline{N}$$

and

$$C_*(X, \underline{N})_{\mathcal{M}(\mathcal{F})} = D_*(X) \otimes_{\mathcal{M}(\mathcal{F})} \underline{N},$$

giving rise, by passage to homology, to abelian groups which we denote by $H^*_G(X;\overline{N})_{\mathcal{C}(\mathcal{F})}$ and $H^G_*(X;\underline{N})_{\mathcal{C}(\mathcal{F})}$, respectively, with $\mathcal{C}=\mathcal{B}$ or $\mathcal{M}$. (Note that the boundary operators are the natural ones, $\partial_*\otimes 1$ and $\mathrm{Hom}(\partial_*,1)$.)

PROPOSITION 3.1. *Let X be a G-CW complex with G-orbits of the form G/H for $H\in\mathcal{F}$. Then there exist natural isomorphisms*

$$\phi\colon H^*_G(X;\overline{N})_{\mathcal{C}(\mathcal{F})}\cong H^*_G(X;\overline{N}),$$

where the second expression denotes Bredon-Illman cohomology with coefficients in $\overline{N}$ (regarded as a Bredon functor).

PROOF. That the theories $H^*_G(X;\overline{N})_{\mathcal{C}(\mathcal{F})}$ are, in fact, equivariant cohomology theories follows at once from the projectivity of the $\mathcal{C}(\mathcal{F})$-functors $C_*(X)$ and $D_*(X)$ in their respective categories. In order to construct the homomorphisms ϕ, it suffices to observe that both $C^*(X;\overline{N})_{\mathcal{B}(\mathcal{F})}$ and $C^*(X;\overline{N})_{\mathcal{M}(\mathcal{F})}$ are sums of copies of $\overline{N}(G/H_\gamma)$ with $H_\gamma\in\mathcal{F}$, and one copy for each G-cell of type $G/H_\gamma\times D^*$, and that the coboundary maps agree with those for Bredon-Illman cohomology on summands. This, in fact, gives an isomorphism on the cochain level, and hence the result. □

Let $E\mathcal{F}$ be a G-space with orbit-types G/H for $H\in\mathcal{F}$ such that $E\mathcal{F}^H$ is contractible for each such H. (These are the universal G-spaces first considered by Palais.) In the next section, we shall show, via the use of Proposition 3.1, that its Bredon cohomology is expressible in terms of Ext functors associated with the category $\mathcal{C}(\mathcal{F})$.

REMARKS 3.2. One could define a chain complex $C_*(X)$ in the category of $\mathcal{F}$-Hecke functors by setting

$$C_n(X)(G/H)=\mathrm{colim}[\Sigma^{n+r}G/H_+,\Sigma^r X^n/X^{n-1}]^G,$$

thereby obtaining a theory $H^*_G(X;\overline{N})_{\mathcal{H}(\mathcal{F})}$ for $\mathcal{F}$-Hecke functors $\overline{N}$. The argument in Proposition 3.1, however, shows that this is again $H^*_G(X;\overline{N})$ if X has only orbit-types given by $\mathcal{F}$. One might have expected that $H^*_G(X;\overline{N})_{\mathcal{H}(\mathcal{F})}$ be an invariant on the category $h\mathcal{T}^G$. That it is not (being one on $hG\mathcal{T}$ instead) results from problems in the cellular theory for this cateogry. Indeed, the category $h\mathcal{T}^G$ misbehaves in the sense that if one has a commuting diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \cup & & \cup \\ A & \xrightarrow{g} & A' \end{array}$$

of G-spaces with $f\in h\mathcal{T}^G$, then g need not be in $h\mathcal{T}^G$. Thus a cellular map $X\to Y$ in $h\mathcal{T}^G$ need not induce a map $X^n/X^{n-1}\to Y^n/Y^{n-1}$ in $h\mathcal{T}^G$.

4. Universal (Palais) G-spaces. In [**W1**] the following is shown.

PROPOSITION 4.1. *Let G be finite. Then $H^*(\hat{\mathbf{Z}};\overline{T})_{\mathcal{H}(\mathcal{F})}\cong H^*_G(E\mathcal{F};\overline{T})$, Bredon equivariant cohomology of $E\mathcal{F}$.*

Here, $H^*(\hat{\mathbf{Z}};\overline{T})_{\mathcal{H}(\mathcal{F})}=\mathrm{Ext}^*_{\mathcal{H}(\mathcal{F})}(\hat{\mathbf{Z}},\overline{T})$, as considered in §2, and $\mathcal{F}$ is closed under passage to subgroups. The Hecke functor $\overline{T}$ is regarded as a Bredon coefficient system via the natural functor described in §1, and assigns to a subgroup H not in $\mathcal{F}$ the trivial group O.

We wish here to generalize this result, as well as the work in [**W2**], regarding G-covering spaces.

PROPOSITION 4.2. *Let $\mathcal{C} = \mathcal{B}$ or $\mathcal{M}$, and let $\mathcal{F}$ be closed under passage, up to conjugacy, to subgroups. Then there exist isomorphisms, natural in $\overline{S}$ and $\overline{T}$,*

$$H^*_G(E\mathcal{F};\overline{S}) \cong H^*_G(E\mathcal{F};\overline{S})_{\mathcal{B}(\mathcal{F})} \cong \operatorname{Ext}^*_{\mathcal{B}(\mathcal{F})}(\hat{\mathbf{Z}};\overline{S})$$

and

$$H^*_G(E\mathcal{F};\overline{T}) \cong H^*_G(E\mathcal{F};\overline{T})_{\mathcal{M}(\mathcal{F})} \cong \operatorname{Ext}^*_{\mathcal{M}(\mathcal{F})}(\mathcal{M}_G;\overline{T})$$

for $\overline{S} \in F(\mathcal{B})$ and $\overline{T} \in F(\mathcal{M})$.

PROOF. Let $C_*(E\mathcal{F})$ denote the chain complex in $F(\mathcal{C}(\mathcal{F}))$ associated with either category. Then $C_*(E\mathcal{F})(G/H)$ is a projective resolution of $C_*(\text{point})(G/H)$ for each $H \in \mathcal{F}$. Indeed, $C_*(E\mathcal{F})(G/H)$ depends, by the definition of $C_*(E\mathcal{F})$ (and G-cellular approximation), only on the H-equivariant homotopy type of $E\mathcal{F}$ up to chain homotopy. Since $H \in \mathcal{F}$ implies that $E\mathcal{F}$ is H-equivariantly contractible, the assertion follows. It now follows that $C_*(E\mathcal{F})$ is a projective resolution of $C_*(\text{point}) = \mathcal{C}_G$ in the category of $\mathcal{C}(\mathcal{F})$-functors, since each $C_n(E\mathcal{F})$ is a sum of objects of the form $\mathcal{C}_H$ with $H \in \mathcal{F}$.

It suffices now to show that $F(\mathcal{C}(\mathcal{F}))(C_*(E\mathcal{F}), T) \cong F(\mathcal{C})(C_*(E\mathcal{F}), T)$ for any $T \in F(\mathcal{C})$. But both chain complexes are isomorphic with corresponding sums of $T(G/H)$'s, by the proof of Lemma 2.1. The result now follows. □

COROLLARY 4.3. *There exist isomorphisms*

$$\operatorname{Ext}^*_{\mathcal{B}(\mathcal{F})}(\hat{\mathbf{Z}},\overline{S}) \cong \operatorname{Ext}^*_{\mathcal{M}(\mathcal{F})}(\mathcal{M}_G,\overline{S}) \cong \operatorname{Ext}^*_{\mathcal{H}(\mathcal{F})}(\hat{\mathbf{Z}},\overline{S})$$

for any $\mathcal{F}$-Hecke functor $\overline{S}$. If $\overline{T}$ is an $\mathcal{F}$-Mackey functor, then the first isomorphism continues to hold, with $\overline{S}$ replaced by $\overline{T}$.

To end this section, we show how to generalize the work in [**W2**] to the case of a general compact Lie group G and arbitrary coefficient systems.

Let A be a finite group. One defines a (G, A)-covering space $p\colon E \to B$ just as in [**W2**]; isotropy subgroups of G act on fibers via some homomorphism $G_b \to A$. If the fixed sets of B are connected, then the homomorphisms $G_b \to A$ may be taken to be the restriction of some fixed homomorphism $\sigma\colon G \to A$. In general, we refer to a (G, A)-covering space whose isotropy action on fibers arises from $\sigma\colon G \to A$ as a (G, A, σ)-covering space. Such objects are classified by the G-space $B^\sigma_G A = EA/A$, where EA is given a $(G \times A^{\mathrm{opp}})$-action by the rule

$$(g, a)a_0[a_1, \dots, a_n] = \sigma(g)a_0 a[a^{-1}a_1 a, \dots, a^{-1}a_n a]$$

(see [**W2**, §1] for details). That the classification continues to hold for G a compact Lie group follows from the classification theory in [**W3**].

Proceeding as in [**W2**], we denote by ψ_a the homomorphism $G \to A$ given by $g \mapsto a^{-1}\sigma(g)a$ for $a \in A$, and let $H_{\psi_a} \subset G \times A^{\mathrm{opp}}$ be the closed subgroup $\{(h, \psi_a(h)^{-1})\}_{h\in H}$ for $H \subset G$. Let $\mathcal{F}$ be the family of subgroups of $G \times A^{\mathrm{opp}}$ containing conjugacy representatives of H_{ψ_a} for every closed subgroup H and each $a \in A$.

Now replace G by $G \times A^{\mathrm{opp}}$ and consider $\mathrm{Ext}^*_{\mathcal{C}(\mathcal{F})}(\mathcal{C}_G, T)$ where $\mathcal{C} = \mathcal{B}$ or $\mathcal{M}$. Since each $(EA)^{H_{\psi_a}}$ is contractible, one has the following

PROPOSITION 4.4. *The Bredon-Illman cohomology of $B^\sigma_G A$ is given by*

$$H^*_G(B^\sigma_G A; T) \cong \mathrm{Ext}^*_{\mathcal{B}(\mathcal{F})}(\mathcal{B}_G; T) \quad \textit{for any } \mathcal{B}(\mathcal{F}) \textit{ functor } T.$$

Here $\mathcal{F}$ is the family of subgroups of $G \times A^{\mathrm{opp}}$ specified above and T is associated with $\mathcal{F}$ and the ambient group $G \times A^{\mathrm{opp}}$.

PROOF. One verifies that $H^*_G(B^\sigma_G A; T) \cong H_{G\times\overline{A}}(EA; T)$, where $\overline{A} = A^{\mathrm{opp}}$, by proceeding as in the proof of [**W2**, §3.2], but replacing $C_*(X)^{H_\psi}$ there by $C_*(X^{H_\psi})$. $\square$

REMARKS 4.5. The structure of G-CW complexes has resulted in a topological proof of the algebraic result 4.3. It would seem interesting to consider an algebraic proof of this result.

Replacement of Ext^* by Tor_* gives one completely analogous results for homology.

In the case of Bredon functors, the identification $H^*_G(E\mathcal{F}, \overline{S}) = \mathrm{Ext}^*_{\mathcal{B}(\mathcal{F})}(\mathbf{Z}, \overline{S})$ of Proposition 4.2 is studied by J. Slominska in [**S1**], where she gives a spectral sequence converging to these Ext groups.

REFERENCES

[**B1**] G. E. Bredon, *Equivariant cohomology theories*, Lecture Notes in Math., Vol. 34, Springer-Verlag, Berlin and New York, 1967.

[**D1**] T. tom Dieck, *Transformation groups and representation theory*, Lecture Notes in Math., Vol. 766, Springer-Verlag, Berlin and New York, 1979.

[**G1**] J. A. Green, *Axiomatic representation theory for finite groups*, J. Pure Appl. Algebra **1** (1971), 41–47.

[**I1**] S. Illman, *Equivariant singular homology and cohomology*, Mem. Amer. Math. Soc. No. 156 (1975).

[**L1**] G. Lewis, Jr., *Mackey functors*, Syracuse University, in preparation.

[**LMM**] G. Lewis, Jr., J. P. May and J. McClure, *Ordinary* RO(G)*-graded equivariant cohomology*, Bull. Amer. Math. Soc. (N.S.) (1981), 208–212.

[**LMMW**] G. Lewis, Jr., J. P. May, J. McClure and S. Waner, *Ordinary* RO(G)*-graded equivariant cohomology*, in preparation.

[**M1**] F. Matumoto, *On G-CW complexes and a theorem of J. H. C. Whitehead*, J. Fac. Sci. Tokyo **18** (1971/72), 363–376.

[**RS**] K. Roggenkamp and L. Scott, *Hecke actions on Picard groups*, University of Virginia, preprint, 1981.

[**S1**] J. Slominska, *Equivariant Bredon cohomology of classifying spaces of families of subgroups*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **28** (1980), 503–508.

[**W1**] S. Waner, *A generalization of the cohomology of groups*, Proc. Amer. Math. Soc. **85** (1982), 469–474.

[**W2**] ——, *Equivariant covering spaces and cohomology*, Proc. Amer. Math. Soc. **88** (1983), 351–356.

[**W3**] ——, *Equivariant classifying spaces and fibrations*, Trans. Amer. Math. Soc. **258** (1980), 385–402.

[**W4**] ——, G-CW(V) *complexes*, Hofstra University, preprint.

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