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HIGHER WHITEHEAD GROUPS OF CERTAIN BUNDLES OVER SEIFERT MANIFOLDS

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ABSTRACT. Vanishing results for $\mathrm{Wh}_j(\pi_1 M) \otimes R$ ($R = \mathbf{Z}$, $\mathbf{Q}$, or $\mathbf{Z}[1/2]$) are obtained when M is a closed aspherical manifold which is the total space of a bundle over an insufficiently large Seifert manifold with infinite fundamental group of hyperbolic type. Allowable fibers include Riemannian flat manifolds and closed aspherical manifolds with poly-$\mathbf{Z}$ fundamental groups. Corollaries concern the homotopy groups of the group $\mathrm{TOP}(M)$ of self-homeomorphisms of M.

Let R be a subring of the rational numbers, n a nonnegative integer, and N a connected manifold.

HYPOTHESIS $\mathrm{A}(n, R)$. $\mathbf{Z}\pi_1 N$ is a right regular Noetherian ring and $\mathrm{Wh}_j(\pi_1 N) \otimes R = 0$ for $0 \leqslant j \leqslant n$.

N satisfies Hypothesis $\mathrm{A}(\infty, R)$ if N satisfies Hypothesis $\mathrm{A}(n, R)$ for all n. It is known that if $\pi_1 N$ is a poly-$\mathbf{Z}$ group then N satisfies $\mathrm{A}(\infty, \mathbf{Z})$. If $\pi_1 N$ is a Bieberbach group then N satisfies $\mathrm{A}(1, \mathbf{Z})$, $\mathrm{A}(3, \mathbf{Z}[1/2])$, and $\mathrm{A}(\infty, \mathbf{Q})$ **[FH1, N2, N3]**. It is conjectured that if $\pi_1 N$ is a Bieberbach group then N satisfies $\mathrm{A}(\infty, \mathbf{Z})$.

Waldhausen's results on the K-theory of generalized free products **[W]**, especially Corollaries 17.1.3 and 17.2.3, yield the following lemma. Recall that the jth Whitehead group of a group G is the jth homotopy group of a space $\mathrm{Wh}^{\mathbf{Z}}(G)$ and that these corollaries establish homotopy Cartesian squares involving these Whitehead spaces.

LEMMA. *Let M be the total space of a fiber bundle over a compact, connected manifold K with fiber N. Assume that N satisfies Hypothesis* $\mathrm{A}(n, R)$. *If K is a surface other than S^2 or $\mathbf{R}P^2$, of if K is a Haken 3-manifold, then for* $0 \leqslant j \leqslant n$, $\mathrm{Wh}_j(\pi_1 M) \otimes R = 0$.

PROOF. An N-bundle over a surface other than S^2 or $\mathbf{R}P^2$ is built up from copies of $N \times D^2$ by amalgamating along copies of $N \times D^1$ or N-bundles over S^1, while an N-bundle over a Haken 3-manifold is built up from copies of $N \times D^3$ by amalgamating along N-bundles over incompressible surfaces (which are not S^2 or $\mathbf{R}P^2$). The arguments for the two cases are essentially the same.

When the base of the bundle is a surface, the integral group rings for the amalgamating subgroups are $\mathbf{Z}\pi_1 N$ or twisted Laurent extensions of $\mathbf{Z}\pi_1 N$: in either

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case the group ring is right regular Noetherian (and hence coherent). This implies, by Corollary 4.2 of [**W**], that $\mathbf{Z}\pi_1 M$ is right regular coherent when K is a surface, so Corollaries 17.1.3 and 17.2.3 of [**W**] are applicable when K is a surface or Haken 3-manifold.

These corollaries give homotopy Cartesian squares with the Whitehead space of a free product with amalgamations or an HNN extension as the lower right-hand corner of the square. Our hypothesis is that, after tensoring with R, the other three spaces in the square are n-connected. This implies that the lower-hand space in the square is also n-connected after tensoring with R. (To verify 0-connectedness, use the argument on p. 250 of [**W**] based on the Bass-Heller-Swan inclusion $\mathrm{Wh}_0(G) \hookrightarrow \mathrm{Wh}_1(G \times \mathbf{Z})$.) Repeated applications of this n-connectedness observation complete the proof of the Lemma.

PROPOSITION. *Let π be a group and $f\colon \pi \to G$ an epimorphism onto a finite group, R a subring of the rational numbers, and n a nonnegative integer. Suppose that for every hyperelementary subgroup H of G the higher Whitehead groups of $f^{-1}(H)$ satisfy* $\mathrm{Wh}_j(f^{-1}(H)) \otimes R = 0$ *for* $0 \leqslant j \leqslant n$. *Then* $\mathrm{Wh}_j(\pi) \otimes R = 0$ *for* $0 \leqslant j \leqslant n$.

PROOF. By [**W**] there is a long exact sequence for any group Γ:

$$(*) \qquad \to \mathrm{Wh}_{j+1}(\Gamma) \to h_j(B\Gamma; K_{\mathbf{Z}}) \xrightarrow{l_j} K_j(\mathbf{Z}\Gamma) \to \mathrm{Wh}_j(\Gamma) \to \cdots \to \mathrm{Wh}_0(\Gamma) \to 0,$$

where $h_j(\ ;K_{\mathbf{Z}})$ is the generalized homology theory arising from the spectrum $K_{\mathbf{Z}}$ for algebraic K-theory and $B\Gamma$ is the classifying space of Γ. This sequence remains exact when tensored with R. From $(*)$ it follows that the conclusion of the theorem is equivalent to the statement; l_j is an isomorphism for all j such that $0 \leqslant j \leqslant n-1$ and an epimorphism for $j = n$.

If H is a subgroup of G define

$$m_j(H) = h_j\big(Bf^{-1}(H); K_{\mathbf{Z}}\big) \otimes R, \quad k_j(H) = K_j\big(\mathbf{Z}f^{-1}(H)\big) \otimes R,$$

and if H and K are subgroups of G and $g \in G$ are such that $gHg^{-1} \subset K$, let (H, g, K) be the homomorphism $H \to K$ given by conjugation by g.

Given $I = (H, g, K)$, there is an induction map $I_*\colon k_j(H) \to k_j(K)$ ("induced map") and a restriction map $I^*\colon k_j(K) \to k_j(H)$ ("transfer"). There is also an induction map $I_*\colon m_j(H) \to m_j(K)$ corresponding to the map in homology induced by $Bf^{-1}(H) \to Bf^{-1}(K)$ and a restriction map $I^*\colon m_j(K) \to m_j(H)$ corresponding to the homology transfer. According to [**FH2**], $l_j\colon m_j(H) \to k_j(H)$ is natural with respect to induction and restriction, and $k_j(\ )$ is a Frobenius module over Swan's Frobenius functor $G_0(\ ) \otimes R$, where $G_0(H)$ is the Grothendieck group of integral representations of H.

Let C be the collection of hyperelementary subgroups of G. Define $m_j(C) = \bigoplus_{H \in C} m_j(H)$ and $k_j(C) = \bigoplus_{H \in C} k_j(H)$. Consider the following commutative diagrams:

$$\begin{array}{ccc} m_j(C) & \xrightarrow{l_j} & k_j(C) \\ I_*\downarrow & & \downarrow I_* \\ m_j(G) & \xrightarrow{l_j} & k_j(G) \end{array} \qquad \begin{array}{ccc} m_j(C) & \xrightarrow{l_j} & k_j(C) \\ I^*\uparrow & & \uparrow I^* \\ m_j(G) & \xrightarrow{l_j} & k_j(G) \end{array}$$

$$I_* = \sum_{H\in C} (H, e, G)_* \qquad I^* = \prod_{H\in C} (H, e, G)^*$$

Since $G_0(\ )\otimes R$ satisfies hyperelementary induction [**Sw**], it follows from [**Dr**, Proposition 1.2] (also see [**N**, Theorem 6.2.7]) that I_*: $k_j(C)\to k_j(G)$ is surjective and I^*: $k_j(G)\to k_j(C)$ is injective. By [**N**, Lemma 6.2.8] I_*: $m_j(C)\to m_j(G)$ is surjective and I^*: $m_j(G)\to m_j(C)$ is injective. By hypothesis l_j: $m_j(C)\to k_j(C)$ is an isomorphism for $0\leqslant j\leqslant n-1$ and an epimorphism for $j=n$. A diagram chase reveals that l_j: $m_j(G)\to k_j(G)$ is an isomorphism for $0\leqslant j\leqslant n-1$ and an epimorphism for $j=n$, completing the proof of the Proposition.

Let K^3 be one of the insufficiently large Seifert manifolds with three exceptional orbits over S^2 given by the invariants $(b;(0,0,0,0)$: (α_1,β_1), (α_2,β_2), $(\alpha_3,\beta_3))$ in the notation of [**O**]. Let $\pi_1K^3\to Q(\alpha_1,\alpha_2,\alpha_3)$ be the quotient by the image in the fundamental group of any regular fiber: here $Q(\alpha_1,\alpha_2,\alpha_3)$ is the orientation-preserving subgroup of a triangle group.

Main Theorem. *Let M be the total space of a bundle with fiber N and base K^3, where K^3 is one of the Seifert manifolds described above. If $\alpha_1^{-1}+\alpha_2^{-1}+\alpha_3^{-1}<1$ and N satisfies Hypothesis* A(n, R), *then* $\mathrm{Wh}_j(\pi_1 M)\otimes R=0$ *for* $0\leqslant j\leqslant n$.

Proof. This argument is essentially that of [**P**] and relies on the fact that $Q=Q(\alpha_1,\alpha_2,\alpha_3)$ is a hyperbolic triangle group if $\alpha_1^{-1}+\alpha_2^{-1}+\alpha_3^{-1}<1$. In [**P** and **S**] it is shown that Q has an epimorphism h: $Q\to G$ to a nonhyperelementary finite group G. Composition gives an epimorphism f: $\pi_1M\to\pi_1K\to Q\xrightarrow{h} G$.

Let H be a subgroup of G. If the covering space of M corresponding to $f^{-1}(H)$ is an N-bundle over a Haken manifold (i.e. if $h^{-1}(H)$ is not a triangle group), then the Lemma is applicable. As H runs over the hyperelementary subgroups of G, though, some of the $h^{-1}(H)$'s may be triangle subgroups of Q, so this observation is not enough to finish the proof. However, hyperbolic triangle groups contain only finitely many triangle subgroups, so we may induce on the number $t(Q)$ of proper triangle subgroups in Q. If $t(Q)=0$ then no $h^{-1}(H)$ is a triangle group and the Proposition and Lemma show that $\mathrm{Wh}_j(\pi_1M)\otimes R=0$ for $0\leqslant j\leqslant n$. If $t(Q)\geqslant 1$, then for any $h^{-1}(H)$ which is a triangle group, $t(h^{-1}(H))<t(Q)$, so the Proposition and the inductive hypothesis imply $\mathrm{Wh}_j(\pi_1M)\otimes R=0$ for $0\leqslant j\leqslant n$.

Let M and N be as in the Main Theorem. Suppose N satisfies the additional

Hypothesis B. N is a closed aspherical manifold and $S_{\mathrm{TOP}}(N\times I^j,\partial)=0$ for $j+\dim(N)\geqslant 6$, where $S_{\mathrm{TOP}}(N\times I^j,\partial)$ is the structure set of topological surgery [**KS**].

An interesting class of manifolds which satisfy Hypothesis B is the closed aspherical manifolds with torsion-free poly- (finite or infinite cyclic) fundamental group **[FH3]**. This class includes closed flat Riemannian manifolds and closed manifolds with poly-$\mathbf{Z}$ fundamental group. Suppose N satisfies Hypothesis $\mathrm{A}(\infty, Q)$ and B (for example, take N to be a closed Riemannian flat manifold or a closed aspherical manifold with poly-$\mathbf{Z}$ fundamental group). By the Main Theorem, $\mathrm{Wh}_j(\pi_1 M) \otimes \mathbf{Q} = 0$ for all j, and by the main theorem of [S], M will also satisfy Hypothesis B in many cases, including these:

(a) α_1, α_2 and α_3 are all odd, or

(b) an odd prime p divides one of the α's, say α_1, and the group $Q(\alpha_1/p, \alpha_2, \alpha_3)$ is also a hyperbolic group of motions.

Let $\mathrm{TOP}(M)$ be the topological group of self-homeomorphisms of M and let $m = \dim(M)$. Suppose M has the properties established for the examples considered above: M satisfies Hypothesis B and all the Whitehead groups of M vanish when tensored with the rationals. Theorem 4.5(B) of **[FH2]** now yields a computation of the rational homotopy groups $\pi_i(\mathrm{TOP}(M)) \otimes \mathbf{Q}$ for $1 \leqslant i \leqslant \phi_2(m)$, where $\phi_2(m)$ is the stable range for topological pseudoisotopy.

COROLLARY 1. *For* $1 \leqslant i \leqslant \phi_2(m)$,

$$\pi_i(\mathrm{TOP}(M)) \otimes \mathbf{Q} = \begin{cases} \mathrm{center}(\pi_1 M) \otimes \mathbf{Q}, & i = 1, \\ \oplus_{j=1}^{\infty} H_{(i+1)-4j}(M, \mathbf{Q}), & i > 0, m \text{ odd}, \\ 0, & i > 0, m \text{ even}. \end{cases}$$

REMARK. The theorem of Farrell and Hsiang quoted above, while stated for the differentiable category in **[FH2]**, is equally valid in the topological category. If M is smoothable, Corollary 1 is true for the diffeomorphism group $\mathrm{Diff}(M)$ in place of $\mathrm{TOP}(M)$ provided $1 \leqslant i \leqslant \phi_1(m)$, where $\phi_1(m)$ is the stable range for smooth pseudoisotopy.

Now suppose N is smoothable and satisfies Hypotheses $\mathrm{A}(1, \mathbf{Z})$, $\mathrm{A}(3, \mathbf{Z}[1/2])$, and B (again this will be the case if N is a closed flat Riemannian manifold or if N is a closed aspherical manifold with $\pi_1 N$ poly-$\mathbf{Z}$). By the Main Theorem and the main theorem of [S], M satisfies Hypothesis B and

$$0 = \mathrm{Wh}_0(\pi_1 M) = \mathrm{Wh}_1(\pi_1 M) = \mathrm{Wh}_2(\pi_1 M) \otimes \mathbf{Z}[1/2] = \mathrm{Wh}_3(\pi_1 M) \otimes \mathbf{Z}[1/2].$$

The following theorem, which is a consequence of the parametrized surgery theory of **[HS]**, was proved in **[N3]** and applies to M as above:

THEOREM. *Suppose* M^m, $m \geqslant 6$, *is a smoothable closed aspherical manifold satisfying Hypothesis* B *and*

$$\mathrm{Wh}_0(\pi_1 M) = \mathrm{Wh}_1(\pi_1 M) = \mathrm{Wh}_2(\pi_1 M) \otimes \mathbf{Z}[1/2] = \mathrm{Wh}_3(\pi_1 M) \otimes \mathbf{Z}[1/2] = 0.$$

Then

(a) *There is a normal abelian subgroup* $H \subset \pi_0(\mathrm{TOP}(M))$ *consisting entirely of* 2-*torsion such that* $\pi_0(\mathrm{TOP}(M))/H \cong \mathrm{Out}(\pi_1 M)$, *where* $\mathrm{Out}(\pi_1 M)$ *is the group of outer automorphisms of* $\pi_1 M$.

(b) $\pi_1(\mathrm{TOP}(M)) \otimes \mathbf{Z}[1/2] \cong \mathrm{center}(\pi_1 M) \otimes \mathbf{Z}[1/2]$.

REMARK. Part (b) depends on the computation by K. Igusa and R. K. Dennis of the kernel of Igusa's map $\chi\colon \pi_1(P^s_{\text{diff}}(M)) \to \mathrm{Wh}_3(\pi_1 M)$, where $P^s_{\text{diff}}(M)$ is the space of stable smooth pseudoisotopies of M **[DI]**.

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