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# RESTRICTED LIE ALGEBRAS WITH SEMILINEAR $p$-MAPPINGS

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ABSTRACT. Restricted Lie algebras $(L,[p])$ having the property that some power of the $p$-mapping $[p]$ is semilinear are investigated.

**1. Introduction.** Since restricted Lie algebras were first introduced by N. Jacobson there have been a number of publications on the numerous correlations between properties of the $p$-mapping and the structure of the underlying Lie algebra. It is the main objective of this paper to determine those restricted Lie algebras $(L,[p])$ where some power of the $p$-mapping is semilinear with respect to the corresponding power of the Frobenius-homomorphism of the base field. It is well known that abelian restricted Lie algebras belong to this class. Our major result illustrates that, in the finite-dimensional case, nilpotency is a necessary condition for $[p]^n$ to be semilinear for some $n \in \mathbf{N}$. However, the converse does not have general validity. Using our result in combination with a commutativity theorem due to I. N. Herstein we demonstrate that every simple associative $F$-algebra having a $p$-semilinear $p$-map is a field.

Throughout this paper, $F$ is assumed to be a field of positive characteristic $p$. $T$ denotes an indeterminate over $F$.

Given a vector space $V$ over $F$ and a field homomorphism $\sigma$: $F \to F$, a mapping $f$: $V \to V$ is called *semilinear* with respect to $\sigma$ if

$$f(\alpha x + y) = \sigma(\alpha) f(x) + f(y) \quad \forall \alpha \in F, \forall x, y \in V.$$

Let $L$ be a Lie algebra over $F$. We consider the Lie algebra $L \otimes_F F[T]$ and define functions $\alpha_i^{(l)}$ by $\mathrm{ad}^l_{x\otimes T + y \otimes 1}(x \otimes 1) = \sum_{i=0}^{l-1} \alpha_i^{(l)}(x, y) \otimes T^i$ for $l \in N$, $x, y \in L$.

REMARK 1. The following identities can be readily verified:

$$\alpha_i^{(l+1)}(x, y) = \begin{cases} \mathrm{ad}_x\big(\alpha_{i-1}^{(l)}(x, y)\big) + \mathrm{ad}_y\big(\alpha_i^{(l)}(x, y)\big), & 1 \leqslant i \leqslant l-1, \\ \mathrm{ad}_x\big(\alpha_{l-1}^{(l)}(x, y)\big), & i = l, \\ \mathrm{ad}_y\big(\alpha_0^{(l)}(x, y)\big), & i = 0. \end{cases}$$

Using induction on $l$ one can check the following facts:

$$(1.1) \qquad \alpha_{l-1}^{(l)}(x, y) = -\mathrm{ad}_x^l(y).$$

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If $A$ is a commutative $F$-algebra we consider the Lie algebra
(1.2) $L \otimes_F A$ and obtain $\alpha_i^{(l)}(x \otimes a, y \otimes b) = \alpha_i^{(l)}(x, y) \otimes a^{i+1}b^{l-i}$.

Let $L$ be a Lie algebra over $F$. A mapping $[p]$: $L \to L$ is called a *p-mapping* if
(1) $\mathrm{ad}_x^{[p]} = \mathrm{ad}_x^p \ \forall x \in L$,
(2) $(\alpha x)^{[p]} = \alpha^p x^{[p]} \ \forall \alpha \in F, \forall x \in L$,
(3) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$ where the functions $s_i$ are given by

$$\mathrm{ad}_{x\otimes T+y\otimes 1}^{p-1}(x \otimes 1) = \sum_{i=1}^{p-1} i s_i(x, y) \otimes T^{i-1}.$$

Using methods of [**3**, p. 187] it can be inductively verified that given an associative algebra $A$ the identity $(x + y)^{p^n} = x^{p^n} + y^{p^n} + \sum_{i=1}^{p^n-1} s_i^{(n)}(x, y)$ holds, where $\mathrm{ad}_{x\otimes T+y\otimes 1}^{p^n-1}(x \otimes 1) = \sum_{i=1}^{p^n-1} i s_i^{(n)}(x, y) \otimes T^{i-1}$. Note that in contrast with the case $n = 1$ the $s_i$ are not uniquely determined in general. Let $(L, [p])$ be a restricted Lie algebra with restricted universal enveloping algebra $u(L) \supset L$. According to the above we obtain

$$(x + y)^{[p]^n} = x^{[p]^n} + y^{[p]^n} + \sum_{i=1}^{p^n-1} s_i^{(n)}(x, y) \quad \forall x, y \in L.$$

The subsequent well-known result will be needed in the sequel.

LEMMA 1.1. *Let $(L, [p])$ be a restricted Lie algebra over $F$ and suppose that $A$ is a commutative $F$-algebra. Then there is one and only one p-map on $L \otimes_F A$ such that $(x \otimes a)^{[p]} = x^{[p]} \otimes a^p \ \forall x \in L \ \forall a \in A$.*

We conclude this section with the following

REMARK 2. Let $A$ be an associative $F$-algebra. For $l \in \mathbf{N}$ we define functions $\lambda_i^{(l)}$ by $(x \otimes T + y \otimes 1)^l = x^l \otimes T^l + y^l \otimes 1 + \sum_{i=1}^{l-1} \lambda_i^{(l)}(x, y) \otimes T^i$, $l \geqslant 2$, $x, y \in A$. A routine computation verifies the subsequent formulas:

$$\lambda_i^{(l+1)}(x, y) = \begin{cases} x\lambda_{i-1}^{(l)}(x, y) + y\lambda_i^{(l)}(x, y), & 2 \leqslant i \leqslant l - 1, \\ xy^l + y\lambda_1^{(l)}(x, y) & i = 1, \\ yx^l + x\lambda_{l-1}^{(l)}(x, y), & i = 1. \end{cases}$$

Given a commutative $F$-algebra $B$ induction on $l$ proves

(1.3) $$\lambda_i^{(l)}(x \otimes a, y \otimes b) = \lambda_i^{(l)}(x, y) \otimes a^i b^{l-i} \quad \text{in } A \otimes_F B.$$

**2. Conditions for semilinearity.** If $(L, [p])$ is a restricted Lie algebra over $F$ $[p]^n$: $L \to L$ is called semilinear if it is semilinear with respect to the homomorphism $\alpha \mapsto \alpha^{p^n}$.

THEOREM 2.1. *Let $(L, [p])$ be a restricted Lie algebra over $F$. Let $x$, $y$ be elements of $L$ such that $(x \otimes T + y \otimes 1)^{[p]^n} = x^{[p]^n} \otimes T^{p^n} + y^{[p]^n} \otimes 1$. Then $\mathrm{ad}_x^{p^n-1}(y) = 0$.*

PROOF. Consider $u(L \otimes_F F[T])$ the restricted universal enveloping algebra of $L \otimes_F F[T]$. Our assumption entails the validity of $(x \otimes T + y \otimes 1)^{p^n} = x^{p^n} \otimes T^{p^n} + y^{p^n} \otimes 1$ in $u(L \otimes_F F[T])$. From Remark 2 of the preceding section we obtain

$0 = \sum_{i=1}^{p^n-1} \lambda_i^{(p^n)}(x \otimes T, y \otimes 1) = \sum_{i=1}^{p^n-1} \lambda_i^{(p^n)}(x, y) \otimes T^i$. In the last equation we made use of the canonical identification $u(L \otimes_F F[T]) \simeq u(L) \otimes_F F[T]$. Hence $\lambda_i^{(p^n)}(x, y) = 0$, $1 \leqslant i \leqslant p^n - 1$. Note that by definition $\lambda_i^{(p^n)}(x, y) = s_i^{(n)}(x, y)$. In particular, $s_{p^n-1}^{(n)}(x, y)$ vanishes. Consequently,

$$\begin{aligned} \mathrm{ad}_x^{p^n-1}(y) &= -\alpha_{p^n-2}^{(p^n-1)}(x, y) \qquad (\text{cf. } (1.1)) \\ &= s_{p^n-1}^{(n)}(x, y) = 0. \end{aligned}$$

COROLLARY 2.2. *Let $(L,[p])$ be a restricted Lie algebra with center $\mathfrak{Z}(L)$ over $F$ such that $[p]^n$: $L \otimes_F F[T] \to L \otimes_F F[T]$ is semilinear. Then $L^{[p]^n} \subset \mathfrak{Z}(L)$. If $L$ is finite dimensional, then $L$ is nilpotent.*

PROOF. By virtue of Theorem 2.1 we obtain $\mathrm{ad}_x^{[p]^n} = \mathrm{ad}_x^{p^n} = 0$ for every $x \in L$. $L^{[p]^n}$ therefore lies centrally in $L$. In particular, $L$ is ad-nilpotent and Engel's Theorem yields the nilpotency of $L$ in case of $L$ being finite dimensional.

We are now able to give a necessary condition for the semilinearity of the mapping $[p]^n$: $L \to L$.

COROLLARY 2.3. *Let $(L,[p])$ be a restricted Lie algebra over $F$ and let $n$ be a positive integer such that* $\mathrm{card}(F) \geqslant p^n$. *Then the following statements hold*:

(1) *If $x$ and $y$ are elements of $L$ such that*

$$(\alpha x + y)^{[p]^n} = \alpha^{p^n} x^{[p]^n} + y^{[p]^n} \quad \forall \alpha \in F,$$

*then* $\mathrm{ad}_x^{p^n-1}(y) = 0$.

(2) *If $[p]^n$: $L \to L$ is semilinear, then $L^{[p]^n}$ is contained in $\mathfrak{Z}(L)$ and $[p]^t$ is semilinear for $t \geqslant n$.*

(3) *If $[p]^n$ is semilinear and $L$ is finite dimensional then $L$ is nilpotent.*

PROOF. (1) For every $\alpha \in F$ there exists a homomorphism of restricted Lie algebras $f_\alpha$: $L \otimes_F F[T] \to L$ such that $f_\alpha(x \otimes \gamma) = \gamma(\alpha)x$ $\forall x \in L$ $\forall \gamma \in F[T]$. According to our general definition we have $(x \otimes T + y \otimes 1)^{[p]^n} - x^{[p]^n} \otimes T^{p^n} - y^{[p]^n} \otimes 1 = \sum_{i=1}^{p^n-1} s_i^{(n)}(x \otimes T, y \otimes 1) = \sum_{i=1}^{p^n-1} s_i^{(n)}(x, y) \otimes T^i$. Applying $f_\alpha$ to this equation and using our present assumption yields

$$(*) \qquad 0 = \sum_{i=1}^{p^n-1} \alpha^i s_i^{(n)}(x, y) \quad \forall \alpha \in F.$$

Let $(e_j)_{j \in J}$ be a basis of $L$ over $F$ and write $s_i^{(n)}(x, y) = \sum_{j \in J} \alpha_{ij} e_j$, $1 \leqslant i \leqslant p^n - 1$. Put $\gamma_j := \sum_{i=1}^{p^n-1} \alpha_{ij} T^i$ then $(*)$ gives $0 = \sum_{j \in J} \gamma_j(\alpha) e_j$ $\forall \alpha \in F$. Since $\deg \gamma_j \leqslant p^n - 1$ we conclude $\gamma_j = 0$ $\forall j \in J$. This yields $0 = \sum_{j \in J} e_j \otimes \gamma_j = \sum_{i=1}^{p^n-1} s_i^{(n)}(x, y) \otimes T^i$. Consequently, the condition of Theorem 2.1 is satisfied by the pair $(x, y)$ and we obtain the desired result.

(2) The first part of our assertion is an immediate consequence of (1). Let $t$ be an integer $\geqslant n$. We write $[p]^t = [p]^{t-n} \circ [p]^n$. By virtue of our assumption $[p]^n$ is semilinear. Since $[p]^n$ maps $L$ into its center and $[p]^{t-n}|_{\mathfrak{Z}(L)}$ is semilinear our assertion readily follows. The last claim is a direct consequence of Engel's Theorem.

The following example illustrates that the nilpotency of a Lie algebra does not necessarily entail the semilinearity of some power of $p$-mapping (cf. [**4**, p. 97]).

Let $V = \oplus_{i=1}^{p} Fv_i$ be a $p$-dimensional vector space. We view $V$ as an abelian Lie algebra. Consider the endomorphism $f \in \mathrm{End}_F(V)$ which is defined by means of $f(v_i) = v_{i+1}$, $1 \leqslant i \leqslant p-1$, $f(v_p) = 0$. Let $L := V \oplus Ff$ denote the semidirect product of $V$ and $Ff$, i.e. $[\alpha f + v, \beta f + w] = \alpha f(w) - \beta f(v)$ $\forall \alpha, \beta \in F$, $\forall v, w \in V$. Evidently, $L^n = f^{n-1}(V)$ $\forall n \geqslant 2$. ($L^n$ denotes the $n$th component of the lower central series of $L$.) In particular, $L^p = Fv_p$ and $L^{p+1} = 0$. The latter equation implies that $\mathrm{ad}_x^p = 0$ $\forall x \in L$. Applying Theorem 11 of [**3**, p. 190], we conclude that there exists a $p$-mapping $[p]$: $L \to L$ such that $v_i^{[p]} = 0$, $1 \leqslant i \leqslant p-1$, $v_p^{[p]} = v_p$, $f^{[p]} = 0$. Observing the identity $\mathrm{ad}^n_{f\otimes T + v_1 \otimes 1}(f \otimes 1) = -f^n(v_1) \otimes T^{n-1}$ we find $s_i(f, v_1) = 0$, $1 \leqslant i \leqslant p-2$, and $s_{p-1}(f, v_1) = v_p$. Consequently, $(f + v_1)^{[p]} = f^{[p]} + v_1^{[p]} + v_p = v_p$ and $(f + v_1)^{[p]^n} = v_p \neq f^{[p]^n} + v_1^{[p]^n}$ $\forall n \in \mathbf{N}$. Hence $[p]^n$: $L \to L$ is not semilinear.

An element $x \in L$ is called *$p$-nilpotent* if $x^{[p]^n} = 0$ for some $n \in \mathbf{N}$. Note that in the preceding example the set $N(L)$ of $p$-nilpotent elements is not a subspace of $L$. Let $x$ be an element of $L$ and let $(Fx)_p$ denote the $p$-subalgebra of $L$ generated by $x$. Evidently, $(Fx)_p = \sum_{i \geqslant 0} Fx^{[p]^i}$. The element $x$ is called *semisimple* if $x \in (Fx^{[p]})_p$. If $L$ is nilpotent then every semisimple element $x \in L$ lies in the center $\mathfrak{Z}(L)$.

THEOREM 2.4. *Let $(L, [p])$ be a finite-dimensional restricted Lie algebra over an infinite, perfect field. Then the following statements are equivalent*:

(1) *There is an $n \in \mathbf{N}$ such that $[p]^n$ is semilinear.*

(2) *$L$ is nilpotent and $N(L)$ is a subspace of $L$.*

PROOF. (1) $\Rightarrow$ (2). According to Corollary 2.3, $L$ is nilpotent. It is clear that $N(L)$ is stable under the multiplication by scalars. Given $x, y \in N(L)$ we find $t \geqslant n$ such that $x^{[p]^t} = y^{[p]^t} = 0$. Owing to Corollary 2.3 $[p]^t$ is semilinear, hence $x + y \in N(L)$.

(2) $\Rightarrow$ (1). Since $F$ is perfect, every element $x$ of $L$ decomposes into a sum $x = y + z$ with $y$ semisimple and $z$ $p$-nilpotent (cf. [**4**, Theorem V.7.2]).

According to our previous remark, there is a subspace $V \subset \mathfrak{Z}(L)$ consisting of semisimple elements such that $L = V \oplus N(L)$. (Note that the sum of two commuting semisimple elements is semisimple.) Since $N(L)$ is finite dimensional there is an $n \in \mathbf{N}$ such that $N(L)^{[p]^n} = 0$. Let $x_1 = y_1 + z_1$, $x_2 = y_2 + z_2$ be elements of $L$, $y_i \in V$, $z_i \in N(L)$, $1 \leqslant i \leqslant 2$. Due to the inclusion $V \subset \mathfrak{Z}(L)$ we obtain

$$(x_1 + x_2)^{[p]^n} = (y_1 + y_2)^{[p]^n} + (z_1 + z_2)^{[p]^n} = y_1^{[p]^n} + y_2^{[p]^n} = x_1^{[p]^n} + x_2^{[p]^n}.$$

Consequently, $[p]^n$ is semilinear.

B. S. Chwe has shown in [**1**] that if $F$ is algebraically closed then every finite-dimensional restricted Lie algebra $(L, [p])$, having the property that 0 is the only zero of the $p$-map, is abelian. This result does not hold over arbitrary base fields. However, we can infer commutativity from semilinearity.

A restricted Lie algebra $(L, [p])$ is called *$p$-algebraic* if $(Fx)_p$ is finite dimensional for every $x \in L$.

COROLLARY 2.5. *Let $F$ be perfect such that* $\mathrm{card}(F) \geqslant p^n$ *and suppose that* $(L, [p])$ *is a $p$-algebraic restricted Lie algebra such that* (1) $[p]^n$: *$L \to L$ is semilinear* (2) 0 *is the only zero of* $[p]$. *Then $L$ is abelian.*

PROOF. Let $x$ be an element of $L$. Then (2) holds for the finite-dimensional restricted Lie algebra $(Fx)_p$. Since $F$ is perfect $[p]$: $(Fx)_p \to (Fx)_p$ is surjective, thus $x \in L^{[p]}$. Now Corollary 2.3 gives $L = L^{[p]^n} \subset \mathfrak{Z}(L)$.

**3. The associative case.** Every associative $F$-algebra $A$ carries the structure of a restricted Lie algebra, where $[a, b] = ab - ba$ and $a^{[p]} = a^p$. Corollary 2.3 allows us to apply a commutativity theorem due to I. N. Herstein.

THEOREM 3.1. *Let $F$ be a field with* $\operatorname{card}(F) \geqslant p^n$. *Suppose that $A$ is an associative $F$-algebra such that* (1) *$a \mapsto a^{p^n}$ is semilinear*; (2) *$A$ does not possess nontrivial nil ideals. Then $A$ is commutative.*

PROOF. Owing to Corollary 2.3, $a^{p^n}$ lies in the center of $A$ for every element $a$ of $A$. Now apply Theorem 3.2.2 of [**2**].

REMARK. In particular, any simple associative $F$-algebra with semilinear Frobenius homomorphism is a field.

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