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ON A CONJECTURE OF M. S. ROBERTSON

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ABSTRACT. We prove that two classes of univalent functions are equal. This settles a conjecture of M. S. Robertson in the affirmative.

1. Introduction. Recently, M. S. Robertson [3] introduced two classes of univalent functions, $\mathcal{G}$ and $\mathcal{G}^*$, and conjectured that they are equal. In this short note, we prove this conjecture.

First, let us define the classes $\mathcal{G}$ and $\mathcal{G}^*$.

DEFINITION 1. Let $\mathcal{G}$ be the class of all functions f, regular and nonvanishing in $\mathbf{B} = \{z : |z| < 1\}$, with $f(0) = 1$, such that

$$\operatorname{Re}\left\{2z\frac{f'(z)}{f(z)} + \frac{1+z}{1-z}\right\} > 0 \quad \text{for } z \in \mathbf{B}.$$

Note that $1 \in \mathcal{G}$.

Let D be a domain, and let a belong to the closure of D. We say that D is *starlike with respect to a* if for each $z \in D$, every point $tz + (1-t)a$, with $0 < t \leq 1$, belongs to D.

DEFINITION 2. Let $\mathcal{G}^*$ be the class of functions f, regular and univalent in $\mathbf{B}$, with $f(0) = 1$ and $\lim_{r\to 1^-} f(r) = 0$, such that $f(\mathbf{B})$ is starlike with respect to the origin, and $\operatorname{Re}(e^{ia} f) > 0$ for some real number a. Also, let $1 \in \mathcal{G}^*$.

For the sake of clarity, we remind the reader of some familiar definitions which are needed.

DEFINITION 3. Let S^* be the class of all functions f, regular in $\mathbf{B}$, with $f(0) = 0$, such that

$$\operatorname{Re} z\frac{f'(z)}{f(z)} > 0 \quad \text{for } z \in \mathbf{B}.$$

Note that there is no restriction on $f'(0)$ in this definition.

It is known that each $f \in S^*$ is univalent and maps $\mathbf{B}$ onto a domain starlike with respect to the origin.

DEFINITION 4. Let S_g be the class of all functions f which satisfy one of the following conditions:

(a) f is regular and univalent in $\mathbf{B}$, and maps $\mathbf{B}$ onto a domain which contains the origin and is starlike with respect to the origin.

(b) f is of the form

$$f(z) = h(z)\frac{(z-\varsigma)(1-\bar{\varsigma}z)}{z}, \qquad |\varsigma| < 1,$$

where $h \in S^*$.

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The equivalence of conditions (a) and (b) was shown by J. Hummel [**1**]. This will be our main tool for the proof.

2. Proof of the conjecture.

THEOREM. $\mathcal{G}^* = \mathcal{G}$.

Before we give the proof, we remark that the set-inclusion $\mathcal{G} \subset \mathcal{G}^*$ was established by M. Robertson [**3**]. A short proof of this fact will be given as a part of the proof of the theorem.

PROOF OF THEOREM. (a) $\mathcal{G}^* \subset \mathcal{G}$. Suppose that $f \in \mathcal{G}^*$ is not identical to 1. It follows directly from Definition 2 that f^2 is univalent, $\lim_{r\to 1^-} f(r) = 0$, and f maps $\mathbf{B}$ onto a domain starlike with respect to the origin. Let D_n be the domain obtained from the union of the range of f^2 and the open disc centered at the origin and of radius $1/n$. Evidently, each D_n is simply connected. Let f_n be a conformal map from $\mathbf{B}$ onto D_n that satisfies $f_n(0) = 1$ and $\arg f_n'(0) = \arg(f^2)'(0)$. By the Carathéodory Kernel Theorem [**2**, p. 29], $f_n \to f^2$ uniformly on compact subsets of $\mathbf{B}$. From Definition 4, each $f_n \in S_g$. Hence for all n we can write

$$f_n(z) = h_n(z)\frac{(z - z_n)(1 - \overline{z}_n z)}{z}, \qquad |z_n| < 1,$$

where $h_n \in S^*$. It can be verified that

$$f_n'(0) = \frac{1}{2}\frac{h_n''(0)}{h_n'(0)} + (1 + |z_n|^2)h_n'(0).$$

Since $f_n'(0) \to (f^2)'(0) \neq 0$ and $h \in S^*$ gives $|h_n''(0)/h_n'(0)| \leq 4$, $h_n'(0)$ is uniformly bounded for all n. Hence, there exists a sequence of positive integers (n_k) so that (h_{n_k}) converges uniformly on compact subsets to either $h \in S^*$ or zero. The latter case is impossible, otherwise $zf_n \to 0$ uniformly on compact subsets of $\mathbf{B}$ and f will be identical to zero. Suppose that $z_{n_k} \to \varsigma$, with $|\varsigma| \leq 1$ (otherwise we choose a subsequence of (z_{n_k}) that does so). Then we can write

$$f^2(z) = h(z)\frac{(z - \varsigma)(1 - \overline{\varsigma} z)}{z}, \qquad |\varsigma| \leq 1.$$

Since f does not admit zero in $\mathbf{B}$, $|\varsigma| = 1$. Furthermore, since $\lim_{r\to 1^-} f(r) = 0$ and h is bounded away from zero for values of z close to $\partial\mathbf{B}$, $\varsigma = 1$. Therefore,

$$f^2(z) = -h(z)\frac{(1 - z)^2}{z},$$

which yields

$$\mathrm{Re}\left\{2z\frac{f'(z)}{f(z)} + \frac{1+z}{1-z}\right\} = \mathrm{Re}\left\{z\frac{h'(z)}{h(z)}\right\} > 0,$$

and $f \in \mathcal{G}$.

(b) $\mathcal{G} \subset \mathcal{G}^*$. Let $f \in \mathcal{G}$, with f not identical to 1, and let $h(z) = f^2(z)z/(1 - z)^2$. Then by simple calculation we have

$$\mathrm{Re}\left\{z\frac{h'(z)}{h(z)}\right\} = \mathrm{Re}\left\{2z\frac{f'(z)}{f(z)} + \frac{1+z}{1-z}\right\} > 0.$$

So we have

(1) $$f^2(z) = h(z)\frac{(1 - z)^2}{z},$$

where $h \in S^*$, with $h'(0) = 1$.

For every positive integer n, let $r_n = 1 - 1/n$, and let

$$g_n(z) = -\frac{h(z)}{r_n}\frac{(z - r_n)(1 - r_n z)}{z}.$$

From Definition 4, each $g_n \in S_g$. Note that each $g_n(0) = 1$, and

$$g_n'(0) = \frac{h''(0)}{2} - \frac{r_n^2 + 1}{r_n} \to \frac{h''(0)}{2} - 2 = (f^2)'(0) \neq 0, \infty,$$

since $h(z) \neq z/(1-z)^2$; otherwise f is identically 1. Also, note that $g_n \to f^2$ uniformly on compact subsets of $\mathbf{B}$. By Hurwitz's Theorem, this implies the univalence of f^2 in $\mathbf{B}$. Let $\Delta = f^2(\mathbf{B})$, and let $\Delta_n = g_n(\mathbf{B})$. Then by the Carathéodory Kernel Theorem [**2**, p. 29], $\Delta_n \to \Delta$ as $n \to \infty$. Now we show that Δ is a domain starlike with respect to the origin. Let $w \in \Delta$. From the definition of the kernel [**2**, p. 28], there exists a domain U and a positive integer N such that $1, w \in U$ and U is contained in Δ_n for all $n > N$. Let H be the domain consisting of all open-closed segments starting from the origin and ending in U. Since each $g_n \in S_g$, each Δ_n is starlike with respect to the origin. This implies that H is contained in Δ_n for all $n > N$. Hence H is contained in Δ, and consequently Δ is starlike with respect to the origin. Since $0 \notin \Delta$, it is not hard to show that there exists a radial slit from the origin to infinity which does not meet Δ. Hence, the univalence of f^2, and the starlikeness of Δ about the origin lead to the univalence of f, and to the starlikeness of $f(\mathbf{B})$ about the origin; moreover, there exists a real number a such that $\mathrm{Re}(e^{ia} f) > 0$ in $\mathbf{B}$. Since the origin is an accessible boundary point of $f(\mathbf{B})$, there exists ς, with $|\varsigma| = 1$, so that $\lim_{r \to 1^-} f(r\varsigma) = 0$ (see [**2**, p. 277]). Since h is bounded away from zero for values of z near $\partial\mathbf{B}$, (1) implies that $\varsigma = 1$. Therefore, $f \in \mathcal{G}^*$ and the proof is complete.

The author can give an alternative shorter proof to the theorem based on D. Styer [**4**]. However, this proof is quite involved, and was avoided for the sake of clarity.

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