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# GENERALIZATIONS OF TAKETA'S THEOREM ON THE SOLVABILITY OF $M$-GROUPS

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ABSTRACT. The condition of monomiality on the irreducible characters of a finite group can be weakened somewhat, and yet results analogous to Taketa's solvability theorem continue to hold.

**1. Introduction.** An $M$-group is a finite group with the property that every irreducible character is induced from a linear character of some subgroup, and Taketa's theorem asserts that such a group is necessarily solvable. (Taketa's original paper is [**2**] and an exposition can be found in [**1**].)

Now if $\lambda$ is a linear character of a group $H$, then $H/\ker\lambda$ is cyclic. Taketa's theorem may therefore be viewed as giving information about a group $G$ when it is known that every $\chi \in \operatorname{Irr}(G)$ is induced from some $\psi \in \operatorname{Irr}(H)$ where $H \subseteq G$ and $H/\ker\psi$ has known structure. Our principal result is the following.

THEOREM A. *Let $\mathcal{F}$ be a class of groups closed under isomorphism, subgroups and extensions. Let $G$ be a finite group such that for every $\chi \in \operatorname{Irr}(G)$, there exists $H \subseteq G$ and $\psi \in \operatorname{Irr}(H)$ such that $\psi^G = \chi$ and $H/\ker\psi \in \mathcal{F}$. Then $G \in \mathcal{F}$.*

Observe that if we take $\mathcal{F} = \{\text{solvable groups}\}$, this already gives a proper generalization of Taketa's theoem.

We find the following corollary amusing.

THEOREM B. *For each positive integer $n$, there exists a finite set $\mathcal{S}(n)$ of nonabelian simple groups such that if every irreducible character of some finite group $G$ is induced from a character of degree $\leq n$, then every nonabelian composition factor of $G$ is isomorphic to an element of $\mathcal{S}(n)$. In particular, $\mathcal{S}(1) = \emptyset$ and $\mathcal{S}(2) = \{A_5\}$.*

Our final generalization of Taketa's theorem is in a different direction.

THEOREM C. *Let $G$ be a finite group. For each $\chi \in \operatorname{Irr}(G)$, let*

$$S(\chi) = \{\psi \in \operatorname{Irr}(H) | H \subseteq G,\ \psi^G = \chi\}.$$

*Assume that for each $\chi \in \operatorname{Irr}(G)$, g.c.d.$\{\psi(1) | \psi \in S(\chi)\} = 1$. Then $G$ is solvable.*

It is natural to wonder whether or not any non-$M$-groups can satisfy the hypotheses of Theorem C. The author does not know the answer.

The reader will undoubtedly observe the close similarlity between the proofs of our results and that of Taketa's original theorem. In fact, there are no essentially new ideas here.

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**2. Classes of groups.** Theorem A deals with classes $\mathcal{F}$ of groups, where $\mathcal{F}$ satisfies the following conditions.

$$\text{(2.1)}\qquad \begin{array}{ll} \text{(i)} & \text{If } K \cong H \in \mathcal{F}, \text{ then } K \in \mathcal{F}. \\ \text{(ii)} & \text{If } K \subseteq H \in \mathcal{F}, \text{ then } K \in \mathcal{F}. \\ \text{(iii)} & \text{If } N \triangleleft G \text{ and } N \in \mathcal{F} \text{ and } G/N \in \mathcal{F}, \text{ then } G \in \mathcal{F}. \end{array}$$

Examples of classes satisfying (2.1) are solvable groups and $\pi$-groups for a fixed set $\pi$ of primes. Note that such a class is closed under taking subdirect products and thus if $\mathcal{F}$ satisfies (2.1) and $G$ is any finite group, then $G$ has a unique $\mathcal{F}$-residual: a normal subgroup $G^{\mathcal{F}}$ minimal with the property that $G/G^{\mathcal{F}} \in \mathcal{F}$. Note that $(G^{\mathcal{F}})^{\mathcal{F}} \triangleleft G$ and $G/(G^{\mathcal{F}})^{\mathcal{F}} \in \mathcal{F}$ by (2.1)(iii). It follows that $(G^{\mathcal{F}})^{\mathcal{F}} = G^{\mathcal{F}}$.

PROOF OF THEOREM A. Let $N = G^{\mathcal{F}}$. Our object is to show that $N = 1$ and so we assume the contrary. It follows that $N \not\subseteq \ker \chi$ for some $\chi \in \mathrm{Irr}(G)$ and we choose such a $\chi$ with minimum possible degree. By hypothesis, $\chi = \psi^G$ where $\psi \in \mathrm{Irr}(H)$ and $H/\ker \psi \in \mathcal{F}$.

Now consider $\theta = (1_H)^G$. Then $1_G$ is a constituent of $\theta$ and thus every other irreducible constituent has degree $< \theta(1) = |G : H| \leq \chi(1)$. It follows, by the choice of $\chi$, that $N$ is contained in the kernel of every irreducible constituent of $\theta$ and thus

$$N \subseteq \ker \theta = \mathrm{core}_G(H) \subseteq H.$$

Now $H/N \subseteq G/N \in \mathcal{F}$ and thus $H/N \in \mathcal{F}$ and $H^{\mathcal{F}} \subseteq N$. But

$$N/H^{\mathcal{F}} \subseteq H/H^{\mathcal{F}} \in \mathcal{F}$$

and therefore $N^{\mathcal{F}} \subseteq H^{\mathcal{F}}$. Since $N^{\mathcal{F}} = N$, we conclude that $H^{\mathcal{F}} = N$.

We have that $H/\ker \psi \in \mathcal{F}$ and so $H^{\mathcal{F}} \subseteq \ker \psi$. Thus $N \subseteq \ker \psi$ and since $N \triangleleft G$, we conclude that $N \subseteq \ker \psi^G = \ker \chi$. This is a contradiction. □

To prove Theorem B we appeal to Jordan's theorem (see 14.12 of [**1**]).

(2.2) LEMMA. *For each integer $n \geq 1$, there is a finite set $S(n)$ of nonabelian simple groups such that if $G$ is any finite group having a faithful character of degree $\leq n$, then all nonabelian composition factors of $G$ are isomorphic to elements of $S(n)$.*

PROOF. By Jordan's theorem, there is an integer $m$, depending only on $n$, such that if $G$ has a faithful character of degree $\leq n$, then there exists abelian $A \triangleleft G$ with $|G/A| \leq m$. It follows that we can take $S(n)$ to be a set of representatives for the isomorphism classes of simple groups of order $\leq m$. □

We mention that we can take $S(1) = \emptyset$ and $S(2) = \{A_5\}$. (The latter statement follows from Theorem 14.23 of [**1**].)

PROOF OF THEOREM B. Assume that every $\chi \in \mathrm{Irr}(G)$ is induced by some character of degree $\leq n$. Let $\mathcal{F}$ be the class of all those finite groups with the property that every nonabelian composition factor of every subgroup is isomorphic to an element of $S(n)$. It is easy to check that $\mathcal{F}$ satisfies (2.1).

Now if $\chi \in \mathrm{Irr}(G)$, then $\chi = \psi^G$ with $\psi \in \mathrm{Irr}(H)$ and $\psi(1) \leq n$. Then $H/\ker \psi$ has a faithful character of degree $\psi(1)$ (as does every one of its subgroups) and so $H/\ker \psi \in \mathcal{F}$ by Lemma (2.2). It follows that $G \in \mathcal{F}$ by Theorem A and thus all nonabelian composition factors of $G$ are isomorphic to elements of $S(n)$. □

## 3. Theorem C.

PROOF OF THEOREM C. Let $N$ be the solvable residual of $G$ and suppose $N > 1$. As in Theorem A, choose $\chi \in \mathrm{Irr}(G)$ of minimum degree with $N \not\subseteq \ker\chi$.

If $H \subseteq G$ and $\chi = \psi^G$ with $\psi \in \mathrm{Irr}(H)$, then all nonprincipal irreducible constituents of $\theta = (1_H)^G$ have degree $< \chi(1)$ and therefore, by the choice of $\chi$, we have $N \subseteq \ker\theta \subseteq H$.

Now let $\alpha$ be an irreducible constituent of $\chi_N$. Since $N \not\subseteq \ker\chi$ we have $\alpha \neq 1_N$, and because $N = N'$, we can conclude that $\alpha(1) > 1$. By hypothesis, we can choose $H \subseteq G$ and $\psi \in \mathrm{Irr}(H)$ with $\psi^G = \chi$ and $\alpha(1)$ not dividing $\psi(1)$. By the previous paragraph, we have $N \subseteq H$.

Since $\psi^G = \chi$, the irreducible constituents of $\psi_N$ are all $G$-conjugate to $\alpha$, and so have degree $\alpha(1)$. Therefore, $\alpha(1)$ divides $\psi(1)$ and this is a contradiction. $\square$

## REFERENCES

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