

**WEIGHTED NORM INEQUALITIES  
FOR THE HARDY-LITTLEWOOD MAXIMAL OPERATOR  
ON SPACES OF HOMOGENEOUS TYPE**

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**ABSTRACT.** The purpose of this note is to give an adequate Calderón-Zygmund type lemma in order to extend to the general setting of spaces of homogeneous type the  $A_p$  weighted  $L^p$  boundedness for the Hardy-Littlewood maximal operator given by M. Christ and R. Fefferman.

Recently Michael Christ and Robert Fefferman gave in [1] a remarkable proof of the weighted norm inequality for the Hardy-Littlewood maximal function operator in  $\mathbf{R}^n$ ,  $\|Mf\|_{L^p(w)} \leq C_p \|f\|_{L^p(w)}$  when the weight belongs to Muckenhoupt  $A_p$  classes and  $p > 1$ . In [2], A. P. Calderón proved this boundedness property for spaces such that the measure of balls is continuous as a function of the radius. In [3], R. Macías and C. Segovia extended this result to general spaces of homogeneous type (defined below) constructing an adequate quasi-distance. In both cases, the reverse Hölder inequality must be extended to this general setting, while the proof given in [1] does not make use of this property and only depends on an adequate Calderón-Zygmund type lemma, the proof of which for cubes in  $\mathbf{R}^n$  is very simple. The purpose of this note is to obtain a decomposition lemma which allows us to extend the proof of Christ and Fefferman to spaces of homogeneous type.

We now introduce some notation and definitions. Let  $X$  be a set, a nonnegative symmetric function on  $X \times X$  shall be called a quasi-distance if there exists a constant  $K$  such that  $d(x, y) \leq K(d(x, z) + d(z, y))$  for every  $x, y, z \in X$ , and  $d(x, y) = 0$  if and only if  $x = y$ . Let  $\mu$  be a positive measure defined on a  $\sigma$ -algebra of subsets of  $X$  which contains the  $d$ -balls and satisfies the following: there exists a constant  $C$  such that  $0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty$  holds for every  $x \in X$  and  $r > 0$ . We shall say that  $(X, d, \mu)$  is a space of homogeneous type if  $X$  is a set endowed with a quasi-distance and a measure satisfying these conditions. If  $B = B(x, r)$  is a  $d$ -ball in  $X$ , we write  $\tilde{B}$  for  $B(x, 5K^2r)$  and  $\hat{B}$  for  $B(x, 15K^5r)$ . Let  $A$  be such that  $\mu(\hat{B}) \leq A\mu(B)$  for every  $B$ . If  $f$  is a positive measurable function defined on  $X$  and  $E$  a measurable set,  $f(E)$  means  $\int_E f d\mu$ ,  $m_E f = \mu(E)^{-1} \cdot f(E)$  and  $Mf(x) = \sup m_B |f|$ , where the supremum is taken over all balls  $B$  containing  $x$ . By modifying slightly the proof of Theorem (1.2) in Chapter III of [4], we get the following covering lemma.

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LEMMA 1. Let  $E$  be a bounded subset of  $X$  and assume that for each  $x \in E$  there exist  $y(x) \in X$  and  $r(x) > 0$  such that  $x \in B(y(x), r(x))$ . Then, there exists a sequence of disjoint balls  $\{B(y(x_i), r(x_i))\}$  such that  $E \subset \bigcup_{i=1}^{\infty} B(y(x_i), 5K^2r(x_i))$ .

LEMMA 2. Suppose  $\mu(X) = \infty$ . For any nonnegative integrable function  $f$  with bounded support,  $b \geq 2A^3 > 1$  and any  $k \in \mathbf{Z}$  such that  $\Omega_k = \{y \in X: b^{k+1} \geq Mf(y) > b^k\} \neq \emptyset$ , there exists a sequence of balls  $\{B_i^k\}_{i \in \mathbf{N}}$  satisfying

$$(2.1) \Omega_k \subset \bigcup_{i=1}^{\infty} \tilde{B}_i^k.$$

$$(2.2) B_i^k \cap B_j^k = \emptyset \text{ if } i \neq j,$$

(2.3) For every  $B_i^k$ , there exists  $x_i^k \in B_i^k$  such that if  $r_i^k$  is the radius of  $B_i^k$ ,  $r \geq 5K^2r_i^k$ , and  $x_i^k \in B(y, r) = B$ , then  $b^{k+1} \geq Mf(x_i^k) \geq m_{B_i^k}f > b^k \geq m_Bf$ .

(2.4) If  $x \notin \bigcup_{j=k}^{\infty} \bigcup_{i=1}^{\infty} \tilde{B}_i^j$  and  $Mf(x) < \infty$ , then  $Mf(x) \leq b^k$ .

(2.5) Let  $I_j^k = \{(l, n) \in \mathbf{Z} \times \mathbf{N}: l \geq k + 2, \tilde{B}_n^l \cap \tilde{B}_j^k \neq \emptyset\}$  and let  $A_j^k = \bigcup_{(l, n) \in I_j^k} \tilde{B}_n^l$ , then  $2\mu(A_j^k) \leq \mu(B_j^k)$ .

(2.6) Let  $E_j^k = \tilde{B}_j^k - A_j^k$ , then  $2\mu(E_j^k) \geq \mu(\tilde{B}_j^k)$  and  $\mu(X - \bigcup_{k, j} E_j^k) = 0$ . If  $x \in E_j^k$  and  $Mf(x) < \infty$ , then  $Mf(x) \leq b^{k+2}$ .

(2.7) If  $F_j^k = B_j^k - A_j^k$ , then  $\mu(F_j^k) \geq \mu(\tilde{B}_j^k)/2A$  and  $\sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} \chi_{F_j^k}(x) \leq 3$ , where  $\chi_E$  denotes the characteristic function of the set  $E$ .

PROOF. If  $x \in \Omega_k$ , the integrability of  $f$  implies that the set  $R_k(x) = \{r > 0: m_Bf > b^k, x \in B = B(y, r)\}$  is bounded. We can choose  $r(x) \in R_k(x)$  in such a way that if  $r \geq 5K^2r(x)$ , then  $r \notin R_k(x)$ . Thus, there is a point  $y(x) \in X$  such that

$$b^{k+1} \geq Mf(x) \geq m_{B(y(x), r(x))}f > b^k \geq m_{B(y, r)}f,$$

whenever  $r \geq 5K^2r(x)$  and  $x \in B(y, r)$ . The boundedness of the support of  $f$  implies that of  $\Omega_k$ , therefore Lemma 1 can be applied to obtain a sequence  $\{B_i^k\}$  satisfying (2.1)–(2.4). In order to get (2.5), let us first show that if  $l \geq k + 2$ ,  $n \in \mathbf{N}$  and  $\tilde{B}_n^l \cap \tilde{B}_j^k \neq \emptyset$ , then

$$(2.8) \quad \tilde{B}_n^l \subset \hat{B}_j^k,$$

even more  $r_n^l \leq r_j^k$ . Indeed, if we suppose that  $r_n^l > r_j^k$ , then  $\tilde{B}_n^l \subset \hat{B}_n^l$ . The last inequality in (2.3) applied to  $B(y, r) = \hat{B}_n^l$  gives

$$b^k \geq m_{\hat{B}_n^l}f \geq \mu(B_n^l)\mu(\hat{B}_n^l)^{-1}m_{B_n^l}f \geq A^{-1}m_{B_n^l}f,$$

by the third inequality in (2.3) applied to the pair  $(l, n)$ , we have  $A^{-1}m_{B_n^l}f > A^{-1}b^l \geq A^{-1}b^{k+2}$ , which is a contradiction. Now (2.3), (2.2) and (2.8) yield (2.5) in the following way:

$$\begin{aligned} \mu(A_j^k) &\leq \sum_{I_j^k} \mu(\tilde{B}_n^l) \leq A \sum_{I_j^k} b^{-l} \int_{B_n^l} f d\mu \leq A \left( \sum_{l=k+2}^{\infty} b^{-l} \right) \int_{\tilde{B}_j^k} f d\mu \\ &\leq A^2 b^{-k-1} (b-1)^{-1} \mu(B_j^k) m_{\hat{B}_j^k}f \leq \mu(B_j^k)/2. \end{aligned}$$

In order to prove (2.6), let  $x$  be a point such that  $Mf(x) < \infty$ ; then  $x \in \Omega_k$  for some  $k \in \mathbf{Z}$ . By (2.1),  $x \in \tilde{B}_j^k$  for some  $j \in \mathbf{N}$ . Assume that  $x \in A_j^k$ , then there exists  $(l, n) \in I_j^k$  such that  $x \in \tilde{B}_n^l$ , and from (2.3) we obtain

$$Mf(x) \geq m_{\tilde{B}_n^l}f \geq A^{-1}m_{B_n^l}f > A^{-1}b^{k+2} > b^{k+1},$$

which is a contradiction. Thus, the sequence  $\{E_j^k\}$  is a covering of  $\{x: Mf(x) < \infty\}$ . On the other hand, on account of the weak type (1,1) boundedness of the Hardy-Littlewood maximal function operator, the set  $\{x: Mf(x) = \infty\}$  is of measure zero and therefore (2.6) is proved. From (2.2) we see that

$$\sum_{j=1}^{\infty} \chi_{F_j^k}(x) = \chi_{\cup_{j=1}^{\infty} F_j^k}(x) \leq \chi_{\cup_{j=1}^{\infty} E_j^k}(x),$$

for any  $k \in \mathbf{Z}$ . By definition of  $E_j^k$  it follows readily that no point of  $X$  belongs to more than three of the sets  $\cup_{j=1}^{\infty} E_j^k$ . Then

$$\sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} \chi_{F_j^k}(x) \leq \sum_{k=-\infty}^{\infty} \chi_{\cup_{j=1}^{\infty} E_j^k}(x) \leq 3,$$

which is (2.7). This finishes the proof of the lemma.

With this result the argument given in [1] can be adapted to our purposes and we shall check the details. Let  $w$  be a weight function satisfying the  $A_p$  condition

$$\sup_B m_B w [m_B w^{1/(1-p)}]^{p-1} < \infty.$$

As in the euclidean case, the weight  $\sigma = w^{1/(1-p)}$  satisfies  $A_q$  where  $q + p = qp$ ; then both  $w$  and  $\sigma$  satisfy the  $A_{\infty}$  condition, that is

$$\mu(E)\mu(B)^{-1} \leq Cw(E)^{\delta}w(B)^{-\delta},$$

for all  $E \subset B$ , with  $C$  and  $\delta$  positive and independent of  $E$  and  $B$ . Suppose  $\mu(X) = \infty$  and  $f$  as in Lemma 2. Then (2.6) and (2.3) yield

$$\begin{aligned} \int_X (Mf)^p w \, d\mu &\leq b^{2p} \sum_{k,j} (m_{B_j^k} f)^p w(E_j^k) \\ &\leq b^{2p} \sum_{k,j} \left[ \frac{1}{\sigma(B_j^k)} \int_{B_j^k} (f\sigma^{-1})\sigma \, d\mu \right]^p \\ &\quad \times \sigma(E_j^k) \frac{\sigma(\tilde{B}_j^k)}{\sigma(E_j^k)} \left\{ \left[ \frac{\sigma(B_j^k)}{\mu(B_j^k)} \right]^{p-1} \cdot \frac{w(\tilde{B}_j^k)}{\mu(B_j^k)} \right\}. \end{aligned}$$

Applying the  $A_{\infty}$  condition on  $w$  and  $\sigma$  and the  $A_p$  condition on  $w$ , the  $L^p(w \, d\mu)$  norm of  $Mf$  is bounded by

$$C \left\{ \sum_{k,j} \left[ \frac{1}{\sigma(B_j^k)} \int_{B_j^k} (f\sigma^{-1})\sigma \, d\mu \right]^p \sigma(E_j^k) \right\}^{1/p}.$$

Applying (2.7) and the  $A_{\infty}$  condition on  $\sigma$ , this is bounded by

$$C \left\{ \sum_{k,j} \left[ \frac{1}{\sigma(B_j^k)} \int_{B_j^k} (f\sigma^{-1})\sigma \, d\mu \right]^p \int_{F_j^k} \sigma \, d\mu \right\}^{1/p} \leq C \left\{ \int_X [M_{\sigma}(f\sigma^{-1})]^p \sigma \, d\mu \right\}^{1/p},$$

where  $M_\sigma$  is the Hardy-Littlewood maximal function operator on the space of homogeneous type  $(X, d, \sigma d\mu)$ . Then, we have

$$\|Mf\|_{L^p(wd\mu)} \leq C \left\{ \int_X f^p \sigma^{-p+1} d\mu \right\}^{1/p} = C \|f\|_{L^p(wd\mu)}.$$

This completes the proof for the case  $\mu(X) = \infty$ . If  $\mu(X) < \infty$ , let  $Y$  be  $X \times \mathbf{R}$ ,  $\delta: Y \times Y \rightarrow \mathbf{R}^+ \cup \{0\}$  defined by  $\delta((x_1, t_1), (x_2, t_2)) = \max\{d(x_1, x_2), |t_1 - t_2|\}$  and  $\nu = \mu \times \lambda$ , where  $\lambda$  is the Lebesgue measure on  $\mathbf{R}$ ; then  $(Y, d, \nu)$  is a space of homogeneous type with  $\nu(Y) = \infty$ . Given a weight  $w$  satisfying  $A_p$  on  $X$ , then  $W(x, t) = w(x)$  satisfies  $A_p$  on  $Y$ . If  $f$  is a measurable function on  $X$ , define  $F(x, t) = f(x)\chi_{(-2R, 2R)}(t)$  on  $Y$ , where  $R$  is such that  $B(x, R) = X$  for every  $x \in X$ . With these definitions it is clear that  $Mf(x) \leq M_Y F(x, t)$  for all  $t \in (-R, R)$ , where  $M_Y$  is the Hardy-Littlewood maximal function operator on  $Y$ . By these remarks, the result just proved applied to  $M_Y, F$  and  $W_Y$  implies the desired inequality.

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