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NILPOTENCY OF DERIVATIONS. II

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ABSTRACT. The authors recently proved that for a semiprime ring without 2-torsion, a nilpotent derivation must have odd nilpotency. In this paper, we show the intriguing phenomenon that for a semiprime ring with characteristic 2, the nilpotency of a nilpotent derivation must be of the form 2^n. Combining these two results, we show that for a general semiprime ring with no torsion condition, the nilpotency of a nilpotent derivation is either odd or a power of 2.

Let R be a semiprime ring and ∂ a derivation of R. ∂ is said to be nilpotent if $\partial^m R = (0)$ for some positive integer m. The smallest such m is called the nilpotency of ∂. It was proved in [**1**] that the nilpotency of ∂ is always odd if R is 2-torsion free. The purpose of this note is to settle the general case without any torsion condition on R. It is known that the nilpotency of ∂ is either a power of 2 or an odd number. This result will be used in [**2**] to show the invariance of the nilpotency of a derivation on any nonzero ideals.

Let us first consider the complementary case that R is of characteristic 2.

THEOREM 1. *Let R be a semiprime ring of characteristic 2. Suppose ∂ is a derivation of R and $\partial^m = 0$ where $2^k < m \leq 2^{k+1} - 1$ for some positive integer k. Then $\partial^{2^k} = 0$ and hence the nilpotency of ∂ is a power of 2.*

PROOF. We proceed by induction on k. If $k = 1$, then $m = 3$ and $\partial^3 = 0$. For any $x, y \in R$, $0 = \partial^3(\partial xy) = \partial^2 x \partial^2 y$. By replacing y by yx and noting that ∂^2 is also a derivation of R, we obtain $0 = \partial^2 x \partial^2(yx) = \partial^2 x(\partial^2 yx + y\partial^2 x) = \partial^2 xy\partial^2 x$ and hence by the semiprimeness of R, $\partial^2 x = 0$ for all $x \in R$.

We now assume $k > 1$.

Case 1. Suppose $m < 2^{k+1} - 1$. Then $m = \sum_{j=0}^{k} \alpha_j 2^j$ where each α_j is either 0 or 1 and at least one of the α_j's is zero. Let i be the smallest one with $\alpha_i = 0$. If $i = 0$, then $m = 2m_0$ where $m_0 = \sum_{j=1}^{k} \alpha_j 2^{j-1}$ and $2^{k-1} < m_0 \leq 2^k - 1$. Set $\delta = \partial^2$. Then δ is a derivation with $\delta^{m_0} = 0$. By the induction hypothesis, $\delta^{2^{k-1}} = 0$ or $\partial^{2^k} = 0$. Now assume $i > 0$. Then $\alpha_0 = 1$ and $m + 1 = 2^i + \sum_{j=i+1}^{k} \alpha_j 2^j = 2^i n$ where $n = 1 + \sum_{j=i+1}^{k} \alpha_j 2^{j-i}$. Let $\delta = \partial^{2^i}$. Then δ is a derivation of R and $2^{k-i} < n \leq 2^{k-i+1} - 1$. Again by induction hypothesis, $\delta^{2^{k-i}} = 0$ or $\partial^{2^k} = 0$.

Case 2. Suppose $m = 2^{k+1} - 1$. In view of Case 1, we need only show that $\partial^{m-1} = 0$. Suppose to the contrary that $\partial^{m-1} \neq 0$. Then since, for any $x, y \in R$, $0 = \partial^m(\partial^{m-2}xy) = \partial^{m-1}x\partial^{m-1}y$, there exists $a \in \partial^{m-1}R$, $a \neq 0$, such that $\partial a = 0$ and $\partial^{m-1}xa = a\partial^{m-1}y = 0$ for all $x, y \in R$. Let I be the ideal of R generated by $\partial^{m-1}R$ and let $S = \{(s,t) | s$ and t are positive integers such that

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$\exists b \in I$ with $b \neq 0$, $\partial b = 0$, $\partial^s Rb = b\partial^t R = (0)\}$. Clearly $(m-1, m-1) \in S$. Partially order S the following way: $(s_1, t_1) \prec (s_2, t_2)$ iff $s_1 \leq s_2$ and $t_1 \leq t_2$. let (p, q) be a minimal element in S and c be a nonzero element in I such that $\partial c = 0$ and $\partial^p Rc = c\partial^q R = (0)$.

If one of p and q is $\leq 2^k$, say $q \leq 2^k$, then $0 = c\partial^{2^k}(xy) = cx\partial^{2^k}y$ for all $x, y \in R$, and consequently $cI = (0)$. Thus, by the semiprimeness of R, $c = 0$, a contradiction. Hence both p and q are greater than 2^k. For any $x, y \in R$,

$$\begin{aligned} 0 &= \partial^m(\partial^{p-2^k-1}xc\partial^{q-2^k}y) = \partial^{2^k-1}\partial^{2^k}(\partial^{p-2^k-1}xc\partial^{q-2^k}y) \\ &= \partial^{2^k-1}(\partial^{p-1}xc\partial^{q-2^k}y) = \partial^{p-1}xc\partial^{q-1}y. \end{aligned}$$

If $c\partial^{q-1}y = 0$ for all $y \in R$, then $(p, q-1) \in S$, contradicting the minimality of (p, q) in S. Hence $c_0 = c\partial^{q-1}y_0 \neq 0$ for some $y_0 \in R$. Evidently, $c_0 \in I$, $\partial c_0 = 0$ and moreover, $c_0\partial^q y = c\partial^{q-1}y_0\partial^q y = c\partial^2(\partial^{q-1}y_0 y) = 0$ for all $y \in R$. Thus $(p-1, q) \in S$, again contradicting the minimality of (p, q) in S. Therefore $\partial^{m-1} = 0$.

Now we are in a position to prove our second result.

THEOREM 2. *Let R be a semiprime ring and ∂ a nilpotent derivation of R. Then the nilpotency of ∂ is either a power of 2 or an odd number.*

PROOF. Let $I = \{x \in R | 2x = 0\}$. The theorem follows immediately from [**1**] if $I = (0)$. We thus assume that $I \neq (0)$. Note that R/I is a 2-torsion free semiprime ring and the induced derivation $\overline{\partial}$ of R/I defined by $\overline{\partial}(x+I) = \partial x + I$ is nilpotent. According to [**1**] the nilpotency of $\overline{\partial}$ is an odd number, say $2n+1$, i.e., $\partial^{2n+1}R \subseteq I$ but $\partial^{2n}R \not\subseteq I$. On the other hand, I itself is a semiprime ring of characteristic 2 and the restriction of ∂ on I is a nilpotent derivation of I. By Theorem 1 the nilpotency of ∂ on I is a power of 2, say 2^k. We claim that $\partial^{2^k}R \cap I = (0)$. Indeed, for any $a \in I$ and $x \in R$, $0 = \partial^{2^k}(xa) = \partial^{2^k}xa + x\partial^{2^k}a = \partial^{2^k}xa$ by noting that $2a = 0$, i.e. $\partial^{2^k}R \subseteq I^l$, the left annihilator of I. Consequently, $\partial^{2^k}R \cap I \subseteq I^l \cap I$ which is (0) by the semiprimeness of R. Thus $\partial^{2^k}R \cap I = (0)$.

If $2n+1 > 2^k$, then $\partial^{2n+1}R \subseteq \partial^{2^k}R \cap I$ and hence $\partial^{2n+1}R = (0)$. The nilpotency of ∂ is $2n+1$ as $\partial^{2n}R \neq (0)$.

If $2n+1 < 2^k$, then $\partial^{2^k}R \subseteq \partial^{2n+1}R \subseteq I$. It follows that $\partial^{2^k}R = (0)$ and since $(0) \neq \partial^{2^k-1}I \subseteq \partial^{2^k-1}R$, the nilpotency of ∂ is 2^k. This completes the proof.

We should note that in Theorems 1 and 2, the hypothesis that R is semiprime is essential. It can be seen from the following example.

EXAMPLE. Let R be the ring of 6×6 upper triangular matrices over $GF(2)$ and ∂ the inner derivation determined by the matrix $A = E_{12} + E_{23} + E_{34} + E_{45} + E_{56}$, where E_{ij} denotes the unit matrix having 1 at the (i, j)-position and zero elsewhere. One can see easily that $A^6 = 0$, $A^5 \neq 0$ and the nilpotency of ∂ is 6 which is neither a power of 2 not an odd number.

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