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TOPOLOGICAL MINIMAX THEOREMS

MICHAEL A. GERAGHTY AND BOR-LUH LIN[1]

ABSTRACT. Mimimax theorems are given using only topological conditions.

Most mimimax theorems involve linear structure (see references in [**4** and **6**]). Minimax theorems without any linear structure have been obtained by Fan [**4**], Terkelsen [**6**] and Wu [**7**]. In this paper, we shall prove some minimax theorems with only topological conditions. The result of Wu is a consequence of our theorems.

Let X be a compact (Hausdorff) space and let $\mathcal{U}(X)$ be the family of all real-valued upper semicontinuous functions on X, i.e., $\mathcal{U}(X) = C(X, \mathbf{R}^*)$, where $\mathbf{R}^*$ is the set of reals with the topology of upper semicontinuity. We require that the topology on $\mathcal{U}(X)$ have the property that the evaluation function $e_x\colon \mathcal{U}(X) \to \mathbf{R}^*$, where $e_x(f) = f(x)$ for all f in $\mathcal{U}(X)$, be continuous for each x in X. Since the range space $\mathbf{R}^*$ is again the set of reals with the topology of upper semicontinuity, e_x is an upper semicontinuous function from $\mathcal{U}(X)$ to the reals $\mathbf{R}$ with the usual topology. In particular, this will hold for "conjoining" topologies in the sense of Dugundji [**3**, p. 274]. For additional material on the convergence of upper semicontinuous functions and related topologies, see [**2** and **5**] and references cited therein.

DEFINITION. A subset $\mathcal{F}$ of $\mathcal{U}(X)$ is said to be *submaximum* if for any f, g in $\mathcal{F}$ there exists a continuous function $S\colon [0,1] \to \mathcal{F}$ such that $S(0) = f$, $S(1) = g$ and for any $b \in [a,c] \subset [0,1]$, $S(b)(x) \leq \max\{S(a)(x), S(c)(x)\}$ for all $x \in X$.

Observe that the last condition is easily seen to be equivalent to the condition that $L_x^\alpha = \{t : t \in [0,1], S(t)(x) < \alpha\}$ is either connected or empty for all x in X and for all $\alpha \in \mathbf{R}$. This will be satisfied if for all x in X the evaluation mapping from $\mathcal{F}$ to $\mathbf{R}$ defined by $f \mapsto f(x)$ is quasiconvex [**1**] on the set $\mathcal{F}$. Since we assume that e_x is continuous on $\mathcal{U}(X)$, L_x^α is open in $[0,1]$. It is clear that every convex subset in $\mathcal{U}(X)$ is submaximum.

LEMMA 1. *Let X be a compact space and let $\mathcal{F}$ be a submaximum set in $\mathcal{U}(X)$. Suppose that for some $\alpha \in \mathbf{R}$, the set $A_f^\alpha = \{x : x \in X, f(x) \geq \alpha\}$ is either empty or connected for all f in $\mathcal{F}$. Then for any g_0, g_1 in $\mathcal{F}$ such that* $\min\{g_0(x), g_1(x)\} < \alpha$ *for all x in X, there exists an element g in $\mathcal{F}$ such that $g(x) < \alpha$ for all x in X.*

PROOF. Let $S\colon [0,1] \to \mathcal{F}$ be a continuous mapping such that $S(0) = g_0$, $S(1) = g_1$ and $S(b)(x) \leq \max\{S(a)(x), S(c)(x)\}$ for all x in X and for any $b \in [a,c] \subset [0,1]$. For each x in X, let $L_x = \{t : S(t)(x) < \alpha\}$ and $A_x = \{t : S(t)(x) \geq \alpha\}$. As remarked above, L_x is an open interval.

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Now, for each $f \in \mathcal{F}$, let $L_f = \{x : x \in X, f(x) < \alpha\}$ and let $A_f = \{x : x \in X, f(x) \geq \alpha\}$. Since $\min\{g_0(x), g_1(x)\} < \alpha$, $X = L_{g_0} \cup L_{g_1}$. Observe that if $X = L_{g_0}$ (respectively, $X = L_{g_1}$), then we may choose $g = g_0$ (respectively, $g = g_1$). We need only consider the case when A_{g_0} and A_{g_1} are nonempty compact subsets of X. Note that if $x \in A_{g_0}$ then $x \in L_{g_1}$ and so $1 \in L_x$. Similarly, $x \in A_{g_1}$ implies that $0 \in L_x$. Thus we may write, for each x in X,

$$L_x = \begin{cases} (M_x, 1] & \text{if } x \in A_{g_0}, \\ [0, m_x) & \text{if } x \in A_{g_1}, \\ [0, 1] & \text{if } x \in L_{g_0} \cap L_{g_1}. \end{cases}$$

Let $m = \inf_{x \in A_{g_1}} m_x$ and $M = \sup_{x \in A_{g_0}} M_x$. Choose x_n in A_{g_1} such that $\{m_{x_n}\}$ decreases to m, and moreover, such that $\{x_n\}$ converges to some element x_0 in $A_{g_1} \subset L_{g_0}$. Fix k. For any $n > k$, we have $m_{x_n} \leq m_{x_k}$. Hence $m_{x_k} \in A_{x_n}$. It follows that $x_n \in A_{S(m_{x_k})}$, a closed set in X by the upper semicontinuity of $S(m_{x_k})$, for all $n > k$. Thus $x_0 \in A_{S(m_{x_k})}$ for all $k = 1, 2, \ldots$. So $m_{x_k} \in A_{x_0}$ for all k. This implies that $m \in A_{x_0}$. Since $m = \inf_{x \in A_{g_1}} m_x$, we conclude that $A_{x_0} = [m, 1]$ and $L_{x_0} = [0, m)$. Similarly, we obtain an element x_1 in A_{g_0} such that $A_{x_1} = [0, M]$ and $L_{x_1} = (M, 1]$. Now $\bigcap_{x \in X} L_x = [0, m) \cap (M, 1]$. If $M < m$, choose $M < t < m$ and set $g = S(t)$. Since $t \in L_x$ for all $x \in X$, we have $g(x) = S(t)(x) < \alpha$ for all x in X. If $m \leq M$, choose $m \leq t \leq M$. Consider the connected set $A_{S(t)}$. We have L_{g_0} and L_{g_1} open sets, $X = L_{g_0} \cup L_{g_1} \supset A_{S(t)}$, $x_0 \in L_{g_0} \cap A_{S(t)}$ and $x_1 \in L_{g_1} \cap A_{S(t)}$. So from the assumption that $A_{S(t)}$ is connected there must exist some $x \in L_{g_0} \cap L_{g_1} \cap A_{S(t)}$. This implies that $L_x = [0, 1]$ and we must have $x \in L_{S(t)}$. This contradicts $x \in A_{S(t)}$. Q.E.D.

THEOREM 2. *Let X be a compact space and let $\mathcal{F}$ be a submaximum set in $\mathcal{U}(X)$. If for any $f_1, f_2, \ldots, f_n$ in $\mathcal{F}$ and any α in $\mathbf{R}$, the set $\{x : x \in X, f_i(x) \geq \alpha, i = 1, 2, \ldots, n\}$ is either connected or empty, then*

$$\sup_X \inf_{\mathcal{F}} f(x) = \inf_{\mathcal{F}} \sup_X f(x).$$

PROOF. It suffices to show that for any $\alpha \in \mathbf{R}$, if $\sup_X \inf_{\mathcal{F}} f(x) < \alpha$ then $\inf_{\mathcal{F}} \sup_X f(x) < \alpha$. By compactness of X, there exist $f_1, f_2, \ldots, f_n$ in $\mathcal{F}$ such that $\sup_X \inf\{f_1(x), \ldots, f_n(x)\} < \alpha$. It remains to find g in $\mathcal{F}$ such that $g(x) < \alpha$ for all x in X. This is done by induction on n. For $n = 1$, let $g = f_1$. Given $\sup_X \inf\{f_1(x), \ldots, f_n(x)\} < \alpha$, let $X_n = A^{\alpha}_{f_n} = \{x : x \in X, f_n(x) \geq \alpha\}$. Then X_n is compact. Let $\mathcal{F}_n = \{f|_{X_n} : f \in \mathcal{F}\}$. Then $\mathcal{F}_n$ is submaximum in $\mathcal{U}(X_n)$. Apply the induction hypothesis to $(X_n, g_1, \ldots, g_{n-1})$, where $g_i = f_i|_{X_n}$, $i = 1, 2, \ldots, n-1$. Then there exists $g_n \in \mathcal{F}_n$ such that $g_n(x) < \alpha$ for all $x \in X_n$. Let $f \in \mathcal{F}$ be such that $f|_{X_n} = g_n$. Then $\sup_X \min\{f(x), f_n(x)\} < \alpha$. By Lemma 1, there exists $g \in \mathcal{F}$ such that $g(x) < \alpha$ for all x in X. This completes the proof of the theorem. Q.E.D.

COROLLARY 3. *Let X be a compact convex set and let $\mathcal{F}$ be a submaximum set in $\mathcal{U}(X)$. If $A^{\alpha}_f = \{x : x \in X, f(x) \geq \alpha\}$ is convex for all f in $\mathcal{F}$ and for all α in $\mathbf{R}$ then* $\sup_X \inf_{\mathcal{F}} f(x) = \inf_{\mathcal{F}} \sup_X f(x)$.

COROLLARY 4. *Let X be a compact space and let $\mathcal{F}$ be a convex set in $\mathcal{U}(X)$. If for any $f_1, \ldots, f_n$ in $\mathcal{F}$ and any α in $\mathbf{R}$, the set $\{x : x \in X, f_i(x) \geq \alpha, i = 1, 2, \ldots, n\}$ is either connected or empty then* $\sup_X \inf_{\mathcal{F}} f(x) = \inf_{\mathcal{F}} \sup_X f(x)$.

LEMMA 5. *Let X be a compact space. For any $f_1, \ldots, f_n$ in $\mathcal{U}(X)$, the sets $\{x : x \in X, f_i(x) \geq \alpha, i = 1, 2, \ldots, n\}$ are connected or empty for all α in $\mathbf{R}$ if and only if the sets $\{x : x \in X, f_i(x) > \alpha, i = 1, 2, \ldots, n\}$ are either connected or empty for all α in $\mathbf{R}$.*

PROOF. For $f \in \mathcal{U}(X)$ and $\alpha \in \mathbf{R}$, let $U_f^\alpha = \{x : x \in X, f(x) > \alpha\}$.

Assume that $\bigcap_{i=1}^n U_{f_i}^\alpha$ is either connected or empty for all $\alpha \in \mathbf{R}$. Fix $\alpha \in \mathbf{R}$. Let $\{\beta_j\}$ be a sequence increasing to α. Then

$$\bigcap_{i=1}^n A_{f_i}^\alpha = \bigcap_{i=1}^n \bigcap_{j=1}^\infty U_{f_i}^{\beta_j} \subset \bigcap_{j=1}^\infty \overline{\bigcap_{i=1}^n U_{f_i}^{\beta_j}} \subset \bigcap_{j=1}^\infty \bigcap_{i=1}^n A_{f_i}^{\beta_j} = \bigcap_{i=1}^n A_{f_i}^\alpha.$$

Thus $\bigcap_{i=1} A_{f_i}^\alpha = \bigcap_{j=1}^\infty \overline{\bigcap_{i=1}^n U_{f_i}^{\beta_j}}$. Since $\overline{\bigcap_{i=1}^n U_{f_i}^{\beta_j}}$, $j = 1, 2, \ldots$, is a nested sequence of compact connected sets or empty sets, the intersection, $\bigcap_{j=1}^\infty \overline{\bigcap_{i=1}^n U_{f_i}^{\beta_j}}$ is either connected or empty. This shows that $\bigcap_{i=1}^n A_{f_i}^\alpha$ is either connected or empty.

Conversely, assume that $\bigcap_{i=1}^n A_{f_i}^\alpha$ is either connected or empty for all α in $\mathbf{R}$. Fix α in $\mathbf{R}$. Let $\{\beta_j\}$ be a sequence decreasing to α. Then

$$\bigcap_{i=1}^n U_{f_i}^\alpha = \bigcap_{i=1}^n \bigcup_{j=1}^\infty A_{f_i}^{\beta_j} = \bigcup_{j=1}^\infty \bigcap_{i=1}^n A_{f_i}^{\beta_j},$$

which is a nested union of either connected or empty sets. Hence $\bigcap_{i=1}^n U_{f_i}^\alpha$ is either connected or empty. Q.E.D.

COROLLARY 6. *Let X be a compact space and let $\mathcal{F}$ be a submaximum set in $\mathcal{U}(X)$. If for any $f_1, f_2, \ldots, f_n$ in $\mathcal{F}$ and any α in $\mathbf{R}$, the sets $\{x : x \in X, f_i(x) > \alpha, i = 1, 2, \ldots, n\}$ are either connected or empty, then*

$$\sup_X \inf_{\mathcal{F}} f(x) = \inf_{\mathcal{F}} \sup_X f(x).$$

COROLLARY 7 (WU [**7**]). *Let X be a separable compact space and let Y be arcwise connected. If $f : X \times Y \to \mathbf{R}$ is a mapping such that for any $x \in X$, $y \in Y$, the mappings $f_x(y) = f(x, y) = f_y(x)$ are continuous and, moreover, f possesses the two properties:*

(1) *for any y_0, y_1 in Y, there exists a continuous mapping $h : [0, 1] \to Y$ such that $h(0) = y_0$, $h(1) = y_1$ and the set $\{t : f(x, h(t)) \geq \alpha\}$ is either connected or empty for all x in X and for all α in $\mathbf{R}$,*

(2) *for any $y_1, \ldots, y_n$ in Y and any α in $\mathbf{R}$, the set $\{x : x \in X, f(x, y_i) < \lambda, i = 1, 2, \ldots, n\}$ is either connected or empty,*

then $\inf_X \sup_Y f(x, y) = \sup_Y \inf_X f(x, y)$.

Let X be a compact space and let $\mathcal{F}$ be a subset in $\mathcal{U}(X)$. X is said to be *supminimum relative to* $\mathcal{F}$ if for any x, y in X there exists a continuous mapping $S : [0, 1] \to X$ such that $S(0) = x$, $S(1) = y$ and for any any $0 \leq a \leq b \leq c \leq 1$, $f(S(b)) \geq \min\{f(S(a)), f(S(c))\}$ for all f in $\mathcal{F}$. If X is convex, this will be satisfied if for any $f \in \mathcal{F}$, f is a quasiconcave function on X [**1**]. It is clear that if $\mathcal{F}$ is supminimum relative to $\mathcal{F}$, then the set $\{x : x \in X, f_i(x) \geq \alpha, i = 1, 2, \ldots, n\}$ is either connected or empty for all $f_1, \ldots, f_n$ in $\mathcal{F}$ and for all α in $\mathbf{R}$. Thus we obtain the following

COROLLARY 8. *Let X be a compact space and let $\mathcal{F}$ be a submaximum subset in $\mathcal{U}(X)$. If X is supminimum relative to $\mathcal{F}$ then* $\sup_X \inf_{\mathcal{F}} f(x) = \inf_{\mathcal{F}} \sup_X f(x)$.

Finally, let us give a simple example to show that Lemma 1 fails when $\mathcal{F}$ is not submaximum. Let $X = \{1, -1\}$ with the discrete topology. Let $\mathcal{U}(X)$ have the sup norm topology. For each $y \in [-1, 0) \cup (0, 1]$, define $f_y(1) = y$ and $f_y(-1) = -y$. Then $\mathcal{F} = \{f_y : y \in [-1, 0) \cup (0, 1]\}$ is easily seen to be homeomorphic to $[-1, 0) \cup (0, 1]$. It is trivial that $\mathcal{F}$ is not submaximum, since there is no path in $\mathcal{F}$ joining f_1 and f_{-1}. Clearly the sets $\{x : x \in X, f_y(x) \geq 0\}$ are connected for all f_y in $\mathcal{F}$. However, $\sup_X \min\{f_1(x), f_{-1}(x)\} < 0$ and $\sup_X h(x) > 0$ for all h in $\mathcal{F}$.

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242