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UNIFORM ALGEBRAS AND PROJECTIONS

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ABSTRACT. If M is a closed A-submodule of $C(X)$ where A is a uniform algebra on X which contains a separating family of unimodular functions, and if M is a quotient space of some $C(Y)$, then M is an ideal in $C(X)$. If there is an example of a uniform algebra A on some X such that $A \neq C(X)$ but A is complemented in $C(X)$, then there is such an example with A separable.

1. Statements of results. The following problem was considered by Glicksberg [**1**]: If A is a uniform algebra on X which is (topologically linearly) complemented in $C(X)$, does it follow that $A = C(X)$? He obtained a number of positive results; subsequent work is summarized in Pełczyński's monograph [**2**].

We shall prove two results. The first reduces to the solution of a special case of Glicksberg's problem provided $M = A$, $Y = X$, and the given linear mapping is a projection (=idempotent linear transformation) on $C(X)$; the case $M = A$ can be readily deduced from Corollary 5.3 of [**2**]. The second result says that if the answer to Glicksberg's question is negative for some A, then it is negative for a separable A, a reduction which may prove useful in settling the problem. Recall that a uniform algebra on X is a closed point-separating subalgebra (over $\mathbf{C}$) of $C(X)$ which contains the constant functions.

THEOREM 1. *Let A be a uniform algebra on X which contains enough unimodular functions to separate the points of X, and let M be a closed A-submodule of $C(X)$. If M is the range of a continuous linear mapping from some $C(Y)$, then M is an ideal in $C(X)$.*

THEOREM 2. *Suppose there is a uniform algebra A on some X such that $A \neq C(X)$ but A is complemented in $C(X)$. Then there is a separable uniform algebra $\tilde{A}$ on some $\tilde{X}$ such that $\tilde{A} \neq C(\tilde{X})$ but $\tilde{A}$ is complemented in $C(\tilde{X})$.*

If K is a nonempty closed subset of X then the ideal $I = \{f \in C(X) \colon f(x) = 0\ \forall x \in K\}$ is necessarily complemented in $C(X)$ if X is metrizable. However, if $X = \beta\mathbf{Z}$ the Stone-Čech compactification of the integers $\mathbf{Z}$ and if $K = X \setminus \mathbf{Z}$ then I is not complemented in $C(X)$, that is, c_0 is not complemented in l_∞; indeed, c_0 is not even a continuous linear image of l_∞. Thus Theorem 1 has no obvious converse in the nonmetrizable case.

In Theorem 2, separability of $\tilde{A}$ is equivalent to metrizability of $\tilde{X}$. The proof will exhibit $\tilde{A}$ as a subalgebra of A and $\tilde{X}$ as a quotient space of X; thus $\tilde{A}$ will be antisymmetric if A is. It will be clear that the proof can be adapted to prove variants of the theorem in which, for example, complementedness is replaced by being a continuous linear image of some $C(Y)$.

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2. Proofs. The proof of Theorem 1 is modeled on that of [**2**, Proposition 4.1].

PROOF OF THEOREM 1. Let $K = \bigcap\{f^{-1}(0)\colon f \in M\}$, $I = \{f \in C(X)\colon f(x) = 0\ \forall x \in K\}$. We must show that $M = I$. Suppose $M \neq I$. Then there is $\mu \in M^{\perp}$ not supported entirely in K. Thus there is $g_0 \in M$ not vanishing identically on the support of μ, so $0 \neq g_0\mu \in A^{\perp}$. Let S denote the multiplicative semigroup of unimodular functions in A. By the Stone-Weierstrass theorem there are u, v in S such that $(g_0\mu)(u/v) \neq 0$. For $\nu \in M(X)$ (=regular complex Borel measures on X) let $\nu^* \in M(T)$ (T the unit circle in the complex plane) be $\nu^*(E) = \nu(v^{-1}(E))$, that is, $\int_T f\,d\nu^* = \int_X (f \circ v)\,d\nu$ for $f \in C(T)$. For each $g \in M$,

$$\int_T z^n d(guv^{-1}\mu)^* = \int_X v^n d(guv^{-1}\mu) = \int_X (guv^{n-1})\,d\mu = 0$$

for each positive integer n, so by the F. and M. Riesz theorem, $\Phi g = (guv^{-1}\mu)^*$ lies in $H^1 m$ where m is Lebesgue measure on T and H^1 is the usual Hardy space. Thus Φ maps M into the separable dual H^1m and is absolutely summing [**2**, Definition 0.2]. If Ψ is a continuous linear mapping of $C(Y)$ onto M then $\Phi \circ \Psi$ is absolutely summing, so compact [**2**, Theorem 0.5]; thus by the open mapping theorem Φ is compact, which in turn implies that $a \to \Phi((a \circ v)g_0) = a\Phi g_0$ is compact from the disc algebra to H^1m. This, however, is false: if $0 \neq h \in L^1(m)$ (the role of h being played above by Φg_0 where $\int_T d(\Phi g_0) = \int_X g_0 uv^{-1}\,d\mu \neq 0$) choose integers N_1, N_2 so that $|\int z^{N_1} h\,dm| = 2\varepsilon > 0$ and (using the Riemann-Lebesgue lemma) $|\int z^n h\,dm| < \varepsilon$ whenever $|n| \geq N_2 > |N_1|$; if n_1, n_2 are distinct (positive) integers then

$$\begin{aligned} \|z^{2n_1N_2}hm - z^{2n_2N_2}hm\|_{M(T)} &= \|z^{N_1}hm - z^{2n_2N_2+N_1-2n_1N_2}hm\|_{M(T)} \\ &\geq \left|\int z^{N_1}h\,dm\right| - \left|\int z^{N_1+2(n_2-n_1)N_2}h\,dm\right| > 2\varepsilon - \varepsilon = \varepsilon, \end{aligned}$$

contradicting the alleged compactness.

PROOF OF THEOREM 2. Let P be a continuous projection on $C(X)$ with range A. We build increasing sequences $A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots$ and $B_0 \subset B_1 \subset \cdots \subset B_n \subset \cdots$ of separable closed subalgebras of $C(X)$ with B_n selfadjoint, $A_n \subset B_n \cap A$, and $P(B_n) \subset A_{n+1}$ as follows. Choose $h \in A$ such that $\overline{h} \notin A$. Let A_0 be the smallest closed subalgebra of $C(X)$ that contains h and 1. Then successively let B_n be the closed selfadjoint subalgebra of $C(X)$ generated by A_n, and let A_{n+1} be the closed subalgebra of $C(X)$ generated by $P(B_n)$. Let A' and B' denote the respective closures of $\bigcup A_n$ and $\bigcup B_n$. These are separable closed subalgebras of $C(X)$, B' is selfadjoint while A' is not ($h \in A' \subset A$), $A' \subset B'$ and $P(B') = A'$. If $\tilde{X}$ is obtained from X by collapsing each common set of constancy for A' (equivalently, for B') to a point, then B' and A' become the required $C(\tilde{X})$ and $\tilde{A}$.

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