PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 91, Number 3, July 1984

# A COEFFICIENT PROBLEM OF BOMBIERI CONCERNING UNIVALENT FUNCTIONS

DAOUD BSHOUTY

ABSTRACT. We answer affirmatively a question raised by Bombieri concerning the behaviour of the coefficients of normalized univalent functions near the Koebe function.

**1. Introduction.** Let $S$ denote the class of functions $f(z) = z + a_2z^2 + \cdots$, regular and univalent in $\{z: |z| < 1\}$. Bombieri [**1**] has shown that there exist constants $c_n$ such that

$$|\mathrm{Re}(a_n - n)| \leq c_n \mathrm{Re}(2 - a_2). \tag{A}$$

In [**2**,. Problem 6.3] he posed the question whether there exist constants $d_n$ such that

$$\big|\,|a_n| - n\big| \leq d_n(2 - |a_2|).$$

This question is answered affirmatively and, in fact, we prove the following version:

$$\big|\,|a_n| - n\big| \leq d_n(\mathrm{Re}(2 - a_2)). \tag{B}$$

The equivalence of the last two inequalities is easily shown by noting that if $f(z)$ is in $S$, then the rotation $f_\varphi$ is in $S$, where

$$f_\varphi(z) = e^{-i\varphi} f(e^{i\varphi}z) = z + e^{i\varphi}a_2z^2 + e^{2i\varphi}a_3z^3 + \cdots.$$

Inequalities (A) and (B) are equivalent to the following inequalities:

$$\mathrm{Re}\, a_n > n - c_n(\mathrm{Re}(2 - a_2)), \tag{Aa}$$

$$\mathrm{Re}\, a_n < n + c_n(\mathrm{Re}(2 - a_2)), \tag{Ab}$$

$$|a_n| > n - d_n(\mathrm{Re}(2 - a_2)), \tag{Ba}$$

$$|a_n| < n + d_n(\mathrm{Re}(2 - a_2)). \tag{Bb}$$

These inequalities are interrelated. (Aa) implies (Ba) and (Bb) implies (Ab). (Aa) describes how small $\mathrm{Re}\, a_n$ can be for functions close to the Koebe function, a result of which is of independent interest. (Ab) is less significant since the stronger local Bierberbach conjecture $\mathrm{Re}\, a_n < n$ is known to be true for functions close to the Koebe function. As to (Bb), it is more significant, since the corresponding conjecture, namely $|a_n| < n$ for functions close to the Koebe function, is still unsolved. In fact, Bombieri himself considers this conjecture to be still open (private communication). However, in [**4**, p. 83], there is an incorrect related statement, as the author claims that by a small rotation of the unit disk one may replace $\mathrm{Re}\, a_n$ by $|a_n|$ in the local Bierberbach conjecture. This is certainly false. Indeed, the strongest result of Bombieri stated in [**5**, p. 26] claims:

Received by the editors April 6, 1983.
1980 *Mathematics Subject Classification*. Primary 30C50.

There are constants $\alpha_n$ and $\delta_n$ such that

$$\text{(C)} \qquad \operatorname{Re} a_n < n - \alpha_n(2 - \operatorname{Re} a_2) \quad \text{if } n \text{ even, } |2 - a_2| < \delta_n.$$

If we apply this to a rotation we get

$$|a_n| < n - \alpha_n|2 - \operatorname{Re} a_2 e^{i(\theta+2j\pi)/(n-1)}| \quad \text{if } n \text{ even, } |2 - a_2 e^{i(\theta+2j\pi)(n-1)}| < \delta_n,$$

where $\theta = -\arg a_n$, $j$ natural. In particular,

$$|a_n| < n - \alpha_n(2 - |a_2|) \quad \text{if } n \text{ even, } |2 - a_2 e^{i(\theta+2j\pi)/(n-1)}| < \delta_n.$$

Noting that $\theta$ depends on the function itself, the class of univalent functions for which

$$|2 - a_2 e^{i(\theta+2\pi j)/(n-1)}| < \delta_n, \quad \text{for some natural } j,$$

is only a subset of the class of univalent functions for which $|2 - |a_2|| < \delta_n$. This is why (Bb) is not implied by (C) applied to a rotation of the function.

In our proof of inequality (B), we will not find explicit bounds for $d_n$, although estimates could be obtained from the argument.

**2. Main lemma.** Our approach is a straightforward application of the Löwner differential equation theory. Let $T$ be the set of slit mappings $f(z) = z + a_2 z^2 + \cdots$, which can be imbedded in a Löwner chain $f(z,t) = e^t z + a_2(t) z^2 + \cdots$ in the sense that

$$\text{(1)} \qquad f(z) \equiv f(z,0),$$

where $f(z,t)$ satisfies

$$\text{(2)} \qquad \frac{\partial f(z,t)}{\partial t} = z \frac{\partial f(z,t)}{\partial z} \cdot \frac{1 + k(t)z}{1 - k(t)z} \quad \text{in } |z| < 1, \ t > 0,$$

and $k(t)$ is a continuous function satisfying $|k(t)| = 1$. In the topology of uniform convergence on closed subsets of $|z| < 1$, $T$ is dense in $S$. Since $a_j$ is a continuous functional on $S$, it is enough to prove coefficient inequalities such as (A) and (B) for the class $T$. Often we will make use of the well-known recursion formulae for $a_n(t)$ (see, for example, [**5**, Chapter 6]). We also note the special case $k(t) \equiv -1$ which gives rise to the chain $f(z,t) = e^t z/(1-z)^2 = \sum_{n=1}^{\infty} n e^t z^n$.

Our aim is to estimate $a_n(t)$. We write

$$k(s) = -e^{i\theta(s)}, \qquad -\pi < \theta(s) \le \pi, \quad \theta(s) \text{ piecewise continuous.}$$

Then

$$\text{(3)} \qquad \begin{aligned} \operatorname{Re}(2e^t - a_2(t)) &= 2e^{2t} \int_t^\infty (1 - \operatorname{Cos}\theta(s)) e^{-s}\, ds \\ &= 4e^{2t} \int_t^\infty \operatorname{Sin}^2 \frac{\theta(s)}{2} \cdot e^{-s}\, ds \le e^{2t} \int_t^\infty \theta^2(s) e^{-s}\, ds \end{aligned}$$

and

$$\text{(4)} \qquad \begin{aligned} |\operatorname{Im} a_2(t)| &= \left| 2e^{2t} \int_t^\infty \operatorname{Sin}\theta(s) \cdot e^{-s}\, ds \right| \\ &\le 2e^{2t} \int_t^\infty |\theta(s)| e^{-s/2} \cdot e^{-s/2}\, ds \le 2e^{3t/2} \left( \int_t^\infty \theta^2(s) e^{-s}\, ds \right)^{1/2} \end{aligned}$$

by the Schwarz inequality.

We shall deduce

LEMMA. *Let $n \geq 2$. There exist constants $\alpha_n$ and $\beta_n$ such that*

$$
\begin{aligned}
|\operatorname{Re} a_n(t) - ne^t| &\leq \alpha_n e^{2t} \int_t^\infty \theta^2(s) e^{-s}\, ds, \\
|\operatorname{Im} a_n(t)| &\leq \beta_n e^{3t/2} \left( \int_t^\infty \theta^2(s) e^{-s}\, ds \right)^{1/2}.
\end{aligned} \tag{5}
$$

PROOF. This is certainly true for $n = 2$ with $\alpha_2 = 1$ and $\beta_2 = 2$ by (3) and (4). We proceed by induction. We assume these inequalities are true for $n = 2, 3, \ldots, m-1$. Now

$$a_m(t) = -2e^{mt} \sum_{\nu=1}^{m-1} \nu \int_t^\infty e^{-ms} k_1(s)^{m-\nu} \cdot a_\nu(s)\, ds$$

and

$$me^t = -2e^{mt} \sum_{\nu=1}^{m-1} \nu \int_t^\infty e^{-ms} (-1)^{m-\nu} \cdot \nu e^s\, ds.$$

To estimate the required quantities we first note the following:

$$k(s)^{m-\nu} a_\nu(s) - (-1)^{m-\nu} \nu e^s = (k(s)^{m-\nu} - (-1)^{m-\nu}) a_\nu(s) + (-1)^{m-\nu} (a_\nu(s) - \nu e^s),$$

so that

$$
\begin{aligned}
&\operatorname{Re}(k(s)^{m-\nu} a_\nu(s) - (-1)^{m-\nu} \nu e^s) \\
&\quad = (-1)^{m-\nu} \cdot \Big( (\operatorname{Re} e^{i(m-\nu)\theta(s)} - 1) \cdot \operatorname{Re} a_\nu(s) \\
&\qquad\qquad - \operatorname{Im} e^{i(m-\nu)\theta(s)} \cdot \operatorname{Im} a_\nu(s) + (\operatorname{Re} a_\nu(s) - \nu e^s) \Big),
\end{aligned}
$$

and thus, using (3), (4) and the induction assumption, we have

$$
\begin{aligned}
&|\operatorname{Re}(k(s)^{m-\nu} a_\nu(s) - (-1)^{m-\nu} \nu e^s)| \\
&\quad \leq \frac{(m-\nu)^2}{4} \cdot \theta^2(s) \cdot \left( \nu e^s + \alpha_\nu e^{2s} \int_s^\infty \theta^2(\sigma) e^{-\sigma}\, d\sigma \right) \\
&\qquad + (m-\nu) |\theta(s)| \beta_\nu e^{3s/2} \left( \int_s^\infty \theta^2(\sigma) e^{-\sigma}\, d\sigma \right)^{1/2} + \alpha_\nu e^{2s} \int_s^\infty \theta^2(\sigma) e^{-\sigma}\, d\sigma \\
&\quad \leq m^3 \cdot e^{2s} \left\{ \left[ (1 + \theta^2(s)) \alpha_\nu \int_s^\infty \theta^2(\sigma) e^{-\sigma}\, d\sigma + \theta^2(s) e^{-s} \right] \right. \\
&\qquad\qquad \left. + |\theta(s)| \beta_\nu e^{-s/2} \left( \int_s^\infty \theta^2(\sigma) e^{-\sigma}\, d\sigma \right)^{1/2} \right\}.
\end{aligned}
$$

Also note that

$$\operatorname{Im}(k(s)^{m-\nu} a_\nu(s)) = \operatorname{Re} k(s)^{m-\nu} \cdot \operatorname{Im} a_\nu(s) + \operatorname{Im} k(s)^{m-\nu} \cdot \operatorname{Re} a_\nu(s).$$

Thus, using the same arguments, we deduce

$$\begin{aligned}
&|\mathrm{Im}(k(s)^{m-\nu}\cdot a_\nu(s))| \\
&\quad\le (1+m^2\cdot\theta^2(s))\beta_\nu e^{3s/2}\left(\int_s^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right)^{1/2} \\
&\qquad + m|\theta(s)|\left(\nu e^s + \alpha_\nu e^{2s}\int_s^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right) \\
&\quad\le m^2\cdot e^{3s/2}\left\{(1+\theta^2(s))\beta_\nu\left(\int_s^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right)^{1/2}\right. \\
&\qquad\qquad \left. + |\theta(s)|e^{-s/2}\left(\nu + \alpha_\nu e^s\int_\sigma^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right)\right\}.
\end{aligned}$$

We conclude that

$$\begin{aligned}
&|\mathrm{Re}\,a_m(t) - me^t| \le 2m^3 e^{mt}\sum_{\nu=1}^{m-1}\nu\int_t^\infty e^{(-m+2)s} \\
&\qquad\cdot\left\{\left[(1+\theta^2(s))\alpha_\nu\int_s^\infty \theta^2(\sigma)e^{-\alpha}\,d\sigma + \theta^2(s)e^{-s}\right]\right. \\
&\qquad\qquad \left. + |\theta(s)|e^{-s/2}\beta_\nu\left(\int_s^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right)^{1/2}\right\}ds
\end{aligned}$$

and

$$\begin{aligned}
&|\mathrm{Im}\,a_m(t)| \le 2m^2 e^{mt}\sum_{\nu=1}^{m-1}\nu\int_t^\infty e^{(-m+3/2)s} \\
&\qquad\cdot\left[(1+\theta^2(s))\beta_\nu\left(\int_s^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right)^{1/2}\right. \\
&\qquad\qquad \left. + |\theta(s)|e^{-s/2}\left(\nu + \alpha_\nu e^s\int_s^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right)\right]ds.
\end{aligned}$$

The integrals involved in the first inequality are of the form

$$\int_t^\infty e^{(-m+2)s}\gamma_\nu(\theta)\left(\int_s^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right)ds, \qquad \int_t^\infty e^{(-m+2)s}\theta^2(s)e^{-s}\,ds$$

and

$$\int_t^\infty e^{(-m+2)s}|\theta(s)|e^{-s/2}\left(\int_s^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right)^{1/2}ds,$$

where $\gamma_\nu(\theta) = (1+\theta^2(s))\alpha_\nu$ for $1 \le \nu \le m-1$, all of which are bounded by

$$c_\nu e^{(-m+2)t}\int_t^\infty \theta^2(s)e^{-s}\,ds$$

(here and henceforth, the symbol $c_\nu$ refers to indefinite constants depending on $\nu$). This is easily verified. In the first case note that $\gamma_\nu(\theta) \le c_\nu$ and $m \ge 3$, so

$$\begin{aligned}\int_t^\infty & e^{(-m+2)s}\gamma_\nu \left(\int_s^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right) ds \\ &\le c_\nu \int_t^\infty e^{(-m+2)s}\left(\int_t^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right) ds \\ &\le c_\nu e^{(-m+2)t}\int_t^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma.\end{aligned}$$

In the second case note that $e^{(-m+2)s} \le e^{(-m+2)t}$ for $s \ge t \ge 0$, whereas in the third case

$$\begin{aligned}\int_t^\infty & e^{(-m+2)s}|\theta(s)|e^{-s/2}\left(\int_s^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right)^{1/2} ds \\ &\le \left(\int_t^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right)^{1/2}\int_t^\infty e^{(-m+2)s}|\theta(s)|e^{-s/2}\,ds \\ &\le e^{(-m+2)t}\int_t^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma.\end{aligned}$$

Note in the last step the Schwarz inequality has been applied to the second factor.

The integrals involved in the second inequality are of the form

$$\int_t^\infty e^{(-m+3/2)s}\delta_\nu(\theta)\left(\int_s^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right)^{1/2} ds$$

and

$$\int_t^\infty e^{(-m+3/2)s}|\theta(s)|e^{-s/2}\eta_\nu(s)\,ds,$$

where

$$\delta_\nu(\theta) = (1+\theta^2(s))\beta_\nu \le c_\nu$$

and

$$\eta_\nu(s) = \beta_2\left(\nu + \alpha_\nu e^s\int_s^\infty \theta^2(\sigma)e^{-\sigma}\,d\sigma\right)^{1/2} \le c_\nu,$$

since $\theta^2(s) = O(\varepsilon)$. Both integrals are bounded by

$$c_\nu e^{(-m+3/2)t}\left(\int_t^\infty \theta^2(s)\theta^{-s}\,ds\right)^{1/2}.$$

To check this for the first integral, just increase its value by integrating inside from $t$ instead of from $s$. In the second integral use the Schwarz inequality.

Substituting these estimates in the corresponding inequalities, we obtain the conclusion of the Lemma.

**3. Conclusion.** We are ready now for our final result:

THEOREM. *There exist constants $b_n$, $c_n$, $d_n$ such that for $n \ge 2$,*

$$|a_n - n| \le b_n(\mathrm{Re}(2-a_2))^{1/2}, \tag{6}$$

$$|\mathrm{Re}\,a_n - n| \le c_n(\mathrm{Re}(2-a_2)), \tag{7}$$

$$||a_n| - n| \leq d_n(\mathrm{Re}(2 - a_2)). \tag{8}$$

PROOF. From (1), we have $a_n = a_n(0)$, so

$$\begin{aligned}\mathrm{Re}(2 - a_2) &= 2\int_0^\infty (1 - \mathrm{Cos}\,\theta(s))e^{-s}\,ds \\ &= 4\int_0^\infty \mathrm{Sin}^2\,\frac{\theta(s)}{2}\cdot e^{-s}\,ds \geq \frac{4}{\pi^2}\int_0^\infty \theta^2(s)e^{-s}\,ds.\end{aligned} \tag{9}$$

Also,

$$\begin{aligned}|a_n - n| &\leq |\mathrm{Re}\,a_n - n| + |\mathrm{Im}\,a_n| \\ &\leq \alpha_n\int_0^\infty \theta^2(s)e^{-s}\,ds + \beta_n\left(\int_0^\infty \theta^2(s)e^{-s}\,ds\right)^{1/2} \quad \text{by (5)}, \\ &\leq (\alpha_n\pi + \beta_n)\cdot\left(\int_0^\infty \theta^2(s)e^{-s}\,ds\right)^{1/2} \quad \text{since } \int_0^\infty \theta^2(s)e^{-s}\,ds \leq \pi^2,\end{aligned}$$

which, in conjunction with (9), proves (6). Inequality (7) is now immediate from (5). Inequality (8) is a consequence of the following inequality.

$$\begin{aligned}||a_n| - n| &= \frac{1}{|a_n| + n}\{(\mathrm{Re}\,a_n + n)(\mathrm{Re}\,a_n - n) + (\mathrm{Im}\,a_n)^2\} \\ &\leq |\mathrm{Re}\,a_n - n| + (\mathrm{Im}\,a_n)^2.\end{aligned}$$

Then (6) and (7) imply (8) and the Theorem is proved.

REMARK 1. It can be shown that the power 1/2 in the Theorem cannot be relaxed. For starlike functions the exact power is 1. In this case the exact value for $b_n$ is known [**3**].

## REFERENCES

1. E. Bombieri, *Sull'integrazione approssimata dell' equazione differentiale di Löwner e le sue applicazioni alla teoria delle funzioni univalenti*, Thesis, Milano, 1963.
2. W. K. Hayman, *Research problems in function theory*, Athlone Press, Belfast, 1967.
3. J. A. Hummel, *The coefficients of starlike functions*, Proc. Amer. Math. Soc. **22** (1969), 311–315.
4. I. M. Milin, *Univalent functions and orthonormal systems*, Transl. Math. Mono., Vol. 49, Amer. Math. Soc., Providence, R.I., 1977.
5. Chr. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

DEPARTMENT OF MATHEMATICS, TECHNION-ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA, ISRAEL