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INDECOMPOSABILITY OF IDEALS IN GROUP RINGS

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ABSTRACT. Let H be a subgroup of G and let I be the (two-sided) ideal of $\mathbf{Z}G$ generated by $\omega(\mathbf{Z}H)$. In this note, we show that I is indecomposable as an ideal in $\mathbf{Z}G$. This extends a result of Linnell [1] and simplifies his argument somewhat.

If H is a group, we will denote the augmentation ideal of the integral group ring $\mathbf{Z}H$ by $\omega(\mathbf{Z}H)$. In this very brief note, we prove the following result.

THEOREM. *Let H be a subgroup of G and let I be the (two-sided) ideal of $\mathbf{Z}G$ generated by $\omega(\mathbf{Z}H)$. Then I is indecomposable as an ideal in $\mathbf{Z}G$.*

The case where H is a normal subgroup of G was recently proved by Linnell [**1**]. Our argument is somewhat simpler and, of course, extends to arbitrary subgroups.

PROOF OF THEOREM. Suppose $I = P \oplus Q$ is a decomposition as an ideal of $\mathbf{Z}G$.

First consider the case where H is a torsion subgroup of G and let $h \in H$. Then $h - 1 = p + q$ where $p \in P$, $q \in Q$ and $pq = qp = 0$.

Hence, for some k, $(1 + p + q)^k = h^k = 1$.

Thus, $k(p+q) + \binom{k}{2}(p^2+q^2) + \cdots + (p^k+q^k) = 0$. Since $P \cap Q = 0$, we conclude that $(1+p)^k = (1+q)^k = 1$. Therefore, $1+p$ and $1+q$ are units of finite order in $\mathbf{Z}G$. Since $(1+p)+(1+q) = 1+h$, either $1+p$ or $1+q$ must have a nonzero identity coefficient. By [**3**, Corollary 2.1.3] and the fact that p and q are contained in $\omega(\mathbf{Z}G)$, we conclude that either $1+p = 1$ or $1+q = 1$. Hence $h - 1 \in P$ or $h - 1 \in Q$. Because $(h_1 - 1)(h_2 - 1) \neq 0$ if $h_1 \neq h_2$ and $h_1, h_2 \neq 1$, we conclude that $I = P$ or $I = Q$, and we are done.

When H is not torsion, we copy part of the argument in [**1**]. Let F be the torsion subgroup of the finite conjugate subgroup of G and let π: $\mathbf{Z}G \to \mathbf{Z}F$ be the natural projection. Since $H \not\subseteq F$, we have $\pi(I) = \mathbf{Z}F$. By [**2**, Theorems 4.2.12 and 4.3.16], we conclude that $\mathbf{Z}F = \pi(I) = \pi(P) \oplus \pi(Q)$ with $\pi(P) \neq 0 \neq \pi(Q)$. However, this contradicts the fact that $\mathbf{Z}F$ is indecomposable [**2**].

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