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BOUNDEDNESS OF VECTOR MEASURES WITH VALUES IN THE SPACES L_0 OF BOCHNER MEASURABLE FUNCTIONS

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ABSTRACT. Let $L_0(Z)$ be the F-space of all Bochner measurable functions from a probability space to a Banach space Z. We prove that every countably additive vector measure taking values in $L_0(Z)$ has bounded range. This generalizes a recent result due to M. Talagrand and, independently, N. J. Kalton, N. T. Peck and J. W. Roberts, asserting the same for the case when Z is the space of scalars.

Let (Ω, Σ, P) be a probability measure space and $Z = (Z, \|\cdot\|)$ a Banach space. Recall (see **[1, 2]**) that a function $f\colon \Omega \to Z$ is said to be (Bochner, strongly or P-) measurable if it is the limit P-a.e. of a sequence of Z-valued Σ-simple functions or, equivalently, if there is an $A \in \Sigma$ with $P(A) = 0$ such that (i) $f(\Omega \setminus A)$ is a separable subset of Z and (ii) $f|(\Omega \setminus A)$ is Borel measurable, i.e., $f^{-1}(B) \cap (\Omega \setminus A) \in \Sigma$ for every Borel subset B of Z. As usual, two measurable functions that are equal P-a.e. are identified, and the vector space of all resulting equivalence classes of measurable functions $f\colon \Omega \to Z$ will be denoted $L_0(Z) = L_0(\Omega, \Sigma, P; Z)$. Of course, without loss of generality we may—and will—assume that P is complete (i.e., subsets of P-null sets are in Σ); then condition (ii) above takes a simplier form: f is Borel measurable. $L_0(Z)$ will be considered with the metrizable vector topology of convergence in P-measure; a convenient F-norm **[3]** defining this topology is given by the formula

$$d(f) = \inf\{a > 0\colon P\{\|f\| > a\} \leqslant a\}.$$

Here, and in what follows, if $f \in L_0(Z)$, then $\|f\|$ is the function $\omega \to \|f(\omega)\|$, and $\{\|f\| > a\} = \{\omega \in \Omega\colon \|f(\omega)\| > a\}$. If $Z = \mathbf{R}$, then we write simply L_0 instead of $L_0(\mathbf{R})$; $L_0^+ = \{f \in L_0\colon f \geqslant 0\}$. Evidently, if $f \in L_0(Z)$, then $\|f\| \in L_0$ and $d(f) = d(\|f\|)$. Our purpose is to prove the following

THEOREM. *Let $\mathfrak{X}$ be a ring of subsets of a set X, and let $m\colon \mathfrak{X} \to L_0(Z)$ be a finitely additive vector measure. If for every disjoint sequence (A_n) in $\mathfrak{X}$ the sequence $(m(A_n))$ is bounded, then m is bounded (i.e., its range $m(\mathfrak{X})$ is a bounded subset of $L_0(Z)$).*

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As an immediate consequence of this we have

COROLLARY. *If $\mathcal{X}$ is a σ-field (or σ-ring), then every countably additive vector measure $m: \mathcal{X} \to L_0(Z)$ is bounded.*

This corollary for $Z = \mathbf{R}$ (and hence also for the case when $\dim Z < \infty$) is a quite recent result due independently to M. Talagrand **[6]** and N. J. Kalton, N. T. Peck and J. W. Roberts **[4]**, and answers a question first raised by Ph. Turpin **[7]**.

Our proof of the Theorem is a modification of Talagrand's proof (which is surely more direct than that of **[4]**), and we carry it over following **[6]** very closely. For $\mathcal{X}$ a field of sets, it is based on a sequence of five lemmas strictly corresponding to those in **[6]**, so that the reader may easily compare both proofs. Our Lemma 6 is needed to pass to the case where $\mathcal{X}$ is merely a ring of sets. In order to make the proofs of the lemmas more transparent, we have decided to extract from these proofs, and formulate as sublemmas, some more general facts which were implicitly contained in them. In the Remarks at the end of the paper we extend the Theorem to metrizable locally pseudo-convex spaces Z and arbitrary positive measure spaces (Ω, Σ, μ).

The reader familiar with **[6]** will certainly agree with us that the most serious difficulty one encounters when attempts to pass from the "scalar" L_0 to the "vector" L_0 is to find a suitable substitute for the Paley-Zygmund inequality used in **[6]**. It may be, therefore, somewhat surprising that a much more elementary inequality established in Sublemma 1(a), valid in all Banach spaces, and the resulting "measure of nonboundedness of m on Y", $c(Y)$ (replacing $b(Y)$ of **[6]**), are fully adequate for our purposes.

The reader is referred to **[6]** for an intuitive motivation of the approach used there —and in the present paper—and to **[4]** for more information (and relevant references) on the significance of the fact that L_0-valued measures are bounded.

The following easily verified properties of the F-norm d on L_0 will be needed below.

(d1) If $f, g \in L_0$ and $|f| \leqslant |g|$, then $d(f) \leqslant d(g)$.

(d2) If $f, f_n \in L_0^+$ and $f_n \uparrow f$ (P-a.e.), then $d(f_n) \uparrow d(f)$.

(d3) If $f \in L_0$ and $a = d(f)$, then $P\{\|f\| > a\} \leqslant a \leqslant P\{\|f\| \geqslant a\}$.

(d4) If $f \in L_0$, then $P\{\|f\| \geqslant c\} \geqslant c$ implies $d(f) \geqslant c$; more generally, $P\{\|f\| \geqslant a\} \geqslant b$ implies $d(f) \geqslant \min(a, b)$.

(These properties will also be used for the functions $\|f\|$, where $f \in L_0(Z)$.)

Throughout, $\mathcal{X}$ is a ring of subsets of a set X and $m: \mathcal{X} \to L_0(Z)$ is a finitely additive vector measure. For $Y \in \mathcal{X}$ we denote

$$Y \cap \mathcal{X} = \{A \in \mathcal{X} : A \subset Y\},$$

$$\mathcal{P}(Y) = \text{the set of all finite } \mathcal{X}\text{-partitions of } Y,$$

$$a(Y) = \inf_{t>0} \sup\{d(tm(A)) : A \in Y \cap \mathcal{X}\}.$$

$a(Y)$ is a natural measure of nonboundedness of m on $Y \cap \mathcal{X}$: $a(Y) > 0$ if and only if $m(Y \cap \mathcal{X})$ is *not* bounded in $L_0(Z)$.

I will denote an arbitrary finite set of indices. Instead of, e.g., $\Sigma_{i \in I} f_i$, $\max_{B \in \mathfrak{B}} \|m(B)\|$, we shall usually write $\Sigma_I f_i$, $\max_{\mathfrak{B}} \|m(B)\|$.

In Sublemmas 1(a) and (b), Q denotes the Bernoulli probability on $E = \{-1, 1\}^I$; thus $Q(\{\varepsilon\}) = 2^{-\operatorname{card} I}$ for all $\varepsilon \in E$.

SUBLEMMA 1(a). *For all finite families $(z_i)_{i\in I}$ in Z,*

$$Q\left\{\varepsilon \in E: \left\|\sum_I \varepsilon_i z_i\right\| \geqslant \max_I \|z_i\|\right\} \geqslant \frac{1}{2}.$$

PROOF. Choose j in I so that $\|z_j\| = \max_I \|z_i\|$, and let $E_j^+ = \{\varepsilon \in E: \varepsilon_j = 1\}$, $E_j^- = E \setminus E_j^+$. If $\varepsilon \in E$, then define $z(\varepsilon) = \sum_I \varepsilon_i z_i$ and set $\varepsilon' = (\varepsilon_i')$, where $\varepsilon_i' = -\varepsilon_i$ for $i \neq j$ and $\varepsilon_j' = \varepsilon_j$. Since $z(\varepsilon) + z(\varepsilon') = 2\varepsilon_j z_j$, we have $\|z(\varepsilon)\| \geqslant \|z_j\|$ or $\|z(\varepsilon')\| \geqslant \|z_j\|$. It follows that at least half of the elements ε in E_j^+ (and in E_j^- as well) satisfy $\|z(\varepsilon)\| \geqslant \|z_j\|$. Hence

$$Q\{\varepsilon \in E_j^-: \|z(\varepsilon)\| \geqslant \|z_j\|\} = Q\{\varepsilon \in E_j^+: \|z(\varepsilon)\| \geqslant \|z_j\|\} \geqslant \tfrac{1}{2} Q(E_j^+) = \tfrac{1}{4}$$

from which the required inequality follows immediately.

SUBLEMMA 1(b). *If $f_i \in L_0(Z)$ for $i \in I$ and $\Omega' \in \Sigma$, then there exists $J \subset I$ such that*

$$P\left\{\omega \in \Omega': \left\|\sum_J f_j(\omega)\right\| \geqslant \frac{1}{2} \max_I \|f_i(\omega)\|\right\} \geqslant \frac{1}{4} P(\Omega').$$

In consequence, if $\Omega' = \{\max_I \|f_i\| \geqslant a\}$ for some $a \geqslant 0$, then $P\{\|\sum_J f_j\| \geqslant \frac{1}{2}a\} \geqslant \frac{1}{4} P\{\max_I \|f_i\| \geqslant a\}$ for some $J \subset I$. Applying this to $a = d(\max_I \|f_i\|)$ and using (d3) *and* (d4), *we get*

$$d\left(\sum_J f_j\right) \geqslant \frac{1}{4} d\left(\max_I \|f_i\|\right) \quad \textit{for some } J \subset I.$$

PROOF. First consider the case when $\Omega' = \Omega$. Denote

$$D = \left\{(\omega, \varepsilon) \in \Omega \times E: \left\|\sum_I \varepsilon_i f_i(\omega)\right\| \geqslant \max_I \|f_i(\omega)\|\right\}$$

and, for $\omega \in \Omega$ and $\varepsilon \in E$, let

$$D_\omega = \{\varepsilon \in E: (\omega, \varepsilon) \in D\}, \qquad D^\varepsilon = \{\omega \in \Omega: (\omega, \varepsilon) \in D\}.$$

From Sublemma 1(a) it follows that $Q(D_\omega) \geqslant \frac{1}{2}$ for all $\omega \in \Omega$; hence, by the Fubini theorem,

$$\frac{1}{2} \leqslant \int_\Omega Q(D_\omega)\, dP(\omega) = \int_E P(D^\varepsilon)\, dQ(\varepsilon) \leqslant \max_E P(D^\varepsilon).$$

Therefore, we can find (and fix) $\varepsilon \in E$ such that $P(D^\varepsilon) \geqslant \frac{1}{2}$, i.e.,

$$(*) \qquad P\left\{\left\|\sum_I \varepsilon_i f_i\right\| \geqslant \max_I \|f_i\|\right\} \geqslant \frac{1}{2}.$$

Denote $I^+ = \{i \in I: \varepsilon_i = 1\}$, $I^- = I \setminus I^+$; then $\sum_I \varepsilon_i f_i = \sum_{I^+} f_i - \sum_{I^-} f_i$ and from $(*)$ it follows that the inequality asserted in our sublemma holds either for $J = I^+$ or $J = I^-$.

The general case follows from the above by replacing P by $P' = (P(\Omega'))^{-1} \cdot P|(\Omega' \cap \Sigma)$ and f_i by $f_i|\Omega'$ when $P(\Omega') > 0$; if $P(\Omega') = 0$, the sublemma is trivial.

LEMMA 1. *For every $Y \in \mathcal{X}$, $\mathcal{A} \in \mathcal{P}(Y)$ and $t > 0$ there exists a subfamily $\mathcal{A}'$ of $\mathcal{A}$ such that*

$$d\left(tm\left(\bigcup \mathcal{A}'\right)\right) \geqslant \tfrac{1}{4} d\left(\max_{\mathcal{A}} \|tm(A)\|\right).$$

PROOF. It suffices to apply the final conclusion of Sublemma 1(b) to $f_i = tm(A_i)$, where $\mathcal{A} = \{A_i : i \in I\}$.

DEFINITION. For $Y \in \mathcal{X}$ we define

$$c(t, Y) = \sup\left\{ d\left(\max_{\mathcal{A}} \|tm(A)\|\right) : \mathcal{A} \in \mathcal{P}(Y)\right\} \quad \text{for } t > 0,$$
$$c(Y) = \inf_{t>0} c(t, Y).$$

Note. The quantities $a(Y)$ and $c(Y)$ remain unchanged when m is replaced by αm, where $\alpha \neq 0$.

The next lemma is a direct consequence of Lemma 1.

LEMMA 2. *For each $Y \in \mathcal{X}$, $\frac{1}{4}c(Y) \leqslant a(Y) \leqslant c(Y)$.*

Lemma 3 will be preceded by four sublemmas.

SUBLEMMA 3(a). *If $Y \in \mathcal{X}$ and $\mathcal{A} \in \mathcal{P}(Y)$, then*

$$c(Y) = \inf_{t>0} \sup\left\{ d\left(\max_{\mathcal{C}} \|tm(C)\|\right) : \mathcal{A} \prec \mathcal{C} \in \mathcal{P}(Y)\right\},$$

where $\mathcal{A} \prec \mathcal{C}$ means that the partition $\mathcal{C}$ is a refinement of $\mathcal{A}$, i.e., each $A \in \mathcal{A}$ is the union of those members of $\mathcal{C}$ which are contained in A.

PROOF. Let $s = \operatorname{card} \mathcal{A}$. Choose any $\mathcal{B} \in \mathcal{P}(Y)$ and set $\mathcal{C} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$; clearly $\mathcal{A} \prec \mathcal{C} \in \mathcal{P}(Y)$. If $B \in \mathcal{B}$, then

$$\|m(B)\| \leqslant \sum_{A \in \mathcal{A}} \|m(A \cap B)\| \leqslant s \max_{A \in \mathcal{A}} \|m(A \cap B)\|.$$

Hence $\max_{\mathcal{B}} \|m(B)\| \leqslant s \max_{\mathcal{C}} \|m(C)\|$ and so

$$d\left(\max_{\mathcal{B}} \left\|\frac{t}{s} m(B)\right\|\right) \leqslant d\left(\max_{\mathcal{C}} \|tm(C)\|\right) \quad \text{for all } t > 0.$$

It follows that

$$c\left(\frac{t}{s}, Y\right) \leqslant \sup\left\{ d\left(\max_{\mathcal{C}} \|tm(C)\|\right) : \mathcal{A} \prec \mathcal{C} \in \mathcal{P}(Y)\right\} \leqslant c(t, Y)$$

and by taking the infima over $t > 0$ we get the desired result.

SUBLEMMA 3(b). *Let $(E, \mathcal{E}, Q)$ be another probability space, and let $D \in \Sigma \otimes \mathcal{E}$. For $\omega \in \Omega$ and $\varepsilon \in E$ set $D_\omega = \{\varepsilon \in E : (\omega, \varepsilon) \in D\}$, $D^\varepsilon = \{\omega \in \Omega : (\omega, \varepsilon) \in D\}$, and assume that for some $\Omega' \in \Sigma$ and $\gamma \geqslant 0$ we have $Q(D_\omega) \geqslant 1 - \gamma$ for all $\omega \in \Omega'$. Then if $\delta > 0$ and $H = \{\varepsilon \in E : P(\Omega' \cap D^\varepsilon) \geqslant (1 - \delta)P(\Omega')\}$, then $Q(H) \geqslant 1 - (\gamma/\delta)$. Hence also*

$$Q\{\varepsilon \in E : P(D^\varepsilon) \geqslant (1 - \delta)P(\Omega')\} \geqslant 1 - (\gamma/\delta).$$

PROOF. As in Sublemma 1(b) it is enough to consider the case $\Omega' = \Omega$. In this case we have

$$1-\gamma \leqslant \int_\Omega Q(D_\omega)\,dP(\omega) = \int_E P(D^\varepsilon)\,dQ(\varepsilon) = \left(\int_H + \int_{E\setminus H}\right) P(D^\varepsilon)\,dQ(\varepsilon)$$
$$\leqslant Q(H) + (1-\delta)(1-Q(H)) = \delta Q(H) + 1 - \delta$$

and the desired inequality follows.

Below, if I is a finite set and $0 < r < 1$, then we denote by $Q_r = Q_{I,r}$ the probability measure on $E = \{0,1\}^I$ which is the (card I)-fold product of the measure on $\{0,1\}$ that assigns mass $1-r$ to the point 0 and mass r to the point 1.

SUBLEMMA 3(c). *If $(a_i)_{i\in I}$ is a finite family in $\mathbf{R}$, then*

$$Q_r\left\{\varepsilon \in E \colon \max_I a_i = \max_I (1-\varepsilon_i)a_i\right\} \geqslant 1 - r.$$

PROOF. For some $j \in I$ we have $a_j = \max_I a_i$. If $\varepsilon \in E$ is such that $\varepsilon_j = 0$, then $\max_I(1-\varepsilon_i)a_i = a_j$. But $Q_r\{\varepsilon \in E \colon \varepsilon_j = 0\} = 1 - r$, which proves the sublemma.

SUBLEMMA 3(d). *If $(g_i)_{i\in I}$ is a finite family in L_0^+, $g = \max_I g_i$ and*

$$g_\varepsilon = \max_I (1-\varepsilon_i) g_i \quad \textit{for } \varepsilon = (\varepsilon_i) \in E,$$

then for every $\Omega' \in \Sigma$ and $\beta > 0$,

$$Q_r\{\varepsilon \in E \colon P\{\omega \in \Omega' \colon g(\omega) = g_\varepsilon(\omega)\} \geqslant (1-\beta r)P(\Omega')\} \geqslant 1 - 1/\beta.$$

In particular, if $\Omega' = \{g \geqslant a\}$ where $a \geqslant 0$, then

$$Q_r\{\varepsilon \in E \colon P\{g_\varepsilon \geqslant a\} \geqslant (1-\beta r)P\{g \geqslant a\}\} \geqslant 1 - 1/\beta$$

and hence, for $a = d(g)$, we obtain

$$Q_r\{\varepsilon \in E \colon d(g_\varepsilon) \geqslant (1-\beta r)d(g)\} \geqslant 1 - 1/\beta.$$

PROOF. This follows from Sublemmas 3(c) and 3(b) applied to $D = \{(\omega, \varepsilon) \in \Omega \times E \colon g(\omega) = g_\varepsilon(\omega)\}$, $Q = Q_r$, $\gamma = r$ and $\delta = \beta r$.

LEMMA 3. *Let $Y \in \mathfrak{X}$, $\mathcal{Q} \in \mathfrak{P}(Y)$, $0 < r < 1$, $E = \{0,1\}^{\mathcal{Q}}$ and $Q_r = Q_{\mathcal{Q},r}$. For $\varepsilon = (\varepsilon_A)_{A\in\mathcal{Q}} \in E$ set*

$$A_\varepsilon = \bigcup\{A \in \mathcal{Q} \colon \varepsilon_A = 1\}.$$

Then for every $\beta > 0$,

$$Q_r\{\varepsilon \in E \colon c(Y\setminus A_\varepsilon) \geqslant (1-\beta r)c(Y)\} \geqslant 1 - 1/\beta.$$

PROOF. Fix $t > 0$ and let $\mathcal{Q} \prec \mathcal{C} \in \mathfrak{P}(Y)$. For $A \in \mathcal{Q}$ and $\varepsilon \in E$ define

$$g_A = \max\{\|tm(C)\| \colon C \in \mathcal{C}, C \subset A\}, \quad g = \max_{\mathcal{Q}} g_A = \max_{\mathcal{C}} \|tm(C)\|,$$
$$\mathcal{C}_\varepsilon = \{C \in \mathcal{C} \colon C \subset Y\setminus A_\varepsilon\} \in \mathfrak{P}(Y\setminus A_\varepsilon),$$
$$g_\varepsilon = \max_{\mathcal{Q}} (1-\varepsilon_A) g_A = \max_{\mathcal{C}_\varepsilon} \|tm(C)\|.$$

Then from Sublemma 3(d) we have $Q_r\{\varepsilon \in E \colon d(g_\varepsilon) \geqslant (1-\beta r)d(g)\} \geqslant 1 - 1/\beta$; hence, as $c(t, Y\setminus A_\varepsilon) \geqslant d(g_\varepsilon)$, it follows that

$$Q_r\left\{\varepsilon \in E \colon c(t, Y\setminus A_\varepsilon) \geqslant (1-\beta r)d\left(\max_{\mathcal{C}}\|tm(C)\|\right)\right\} \geqslant 1 - 1/\beta.$$

It is easily seen that this remains valid if $d(\max_{\mathcal{C}}\|tm(C)\|)$ is replaced by $\sup\{d(\max_{\mathcal{C}}\|tm(C)\|): \mathcal{A} \prec \mathcal{C} \in \mathcal{P}(Y)\}$ and hence, using Sublemma 3(a), we get

$$Q_r\{\varepsilon \in E: c(t, Y \setminus A_\varepsilon) \geqslant (1-\beta r)c(Y)\} \geqslant 1 - 1/\beta.$$

The required inequality now follows when $t \downarrow 0$.

LEMMA 4. *For every $Y \in \mathcal{X}$, $q \in \mathbf{N}$ and $t > 0$, there exists $\mathcal{A} \in \mathcal{P}(Y)$ such that*

$$P\{\omega \in \Omega: \operatorname{card}\{A \in \mathcal{A}: \|tm(A)(\omega)\| \geqslant 1\} \geqslant q\} \geqslant \tfrac{1}{5}a(Y).$$

PROOF. The proof when $t = 1$ is the same as for Lemma 4 in [**6**]. From this and the fact that $a(Y)$ is the same for m as for tm, the case of an arbitrary $t > 0$ follows directly.

SUBLEMMA 5. *If $(g_i)_{i \in I}$ is a finite family in L_0^+, $0 < r < 1$, $q \in \mathbf{N}$ and*

$$\Omega' = \{\omega \in \Omega: \operatorname{card}\{i \in I: g_i(\omega) \geqslant 1\} \geqslant q\},$$

then for every $\delta > 0$,

$$Q_r\Big\{\varepsilon \in E: P\Big\{\max_{I_\varepsilon^+} g_i \geqslant 1\Big\} \geqslant (1-\delta)P(\Omega')\Big\} \geqslant 1 - \delta^{-1}(1-r)^q,$$

where $I_\varepsilon^+ = \{i \in I: \varepsilon_i = 1\}$.

PROOF. Let $D = \{(\omega, \varepsilon) \in \Omega \times E: \max_{I_\varepsilon^+} g_i(\omega) \geqslant 1\}$. If $\omega \in \Omega'$ and $J(\omega) = \{i \in I: g_i(\omega) \geqslant 1\}$, then $\operatorname{card} J(\omega) \geqslant q$ and

$$E \setminus D_\omega = \Big\{\varepsilon \in E: \max_{I_\varepsilon^+} g_i(\omega) < 1\Big\} \subset \{\varepsilon \in E: J(\omega) \subset I \setminus I_\varepsilon^+\}.$$

Hence $1 - Q_r(D_\omega) \leqslant (1-r)^{\operatorname{card} J(\omega)} \leqslant (1-r)^q$ and so

$$Q_r(D_\omega) \geqslant 1 - (1-r)^q \quad \text{for all } \omega \in \Omega'.$$

Applying Sublemma 3(b) we are done.

LEMMA 5. *For every $Y \in \mathcal{X}$, $q \in \mathbf{N}$ and $t > 0$, if $\mathcal{A} \in \mathcal{P}(Y)$ is chosen according to Lemma 4, then for $0 < r < 1$ and $Q_r = Q_{\mathcal{A},r}$ we have*

$$Q_r\Big\{\varepsilon \in E: \sup_{B \in A_\varepsilon \cap \mathcal{X}} d(tm(B)) \geqslant \tfrac{1}{40}a(Y)\Big\} \geqslant 1 - 2(1-r)^q,$$

where $A_\varepsilon = \bigcup\{A: \varepsilon_A = 1\}$ for $\varepsilon = (\varepsilon_A)_{A \in \mathcal{A}} \in E = \{0,1\}^{\mathcal{A}}$.

PROOF. Applying Sublemma 5 with $\delta = 1/2$ and denoting $\mathcal{A}_\varepsilon = \{A \in \mathcal{A}: \varepsilon_A = 1\} \in \mathcal{P}(A_\varepsilon)$, we obtain

$$Q_r\Big\{\varepsilon \in E: P\Big\{\max_{\mathcal{A}_\varepsilon}\|tm(A)\| \geqslant 1\Big\} \geqslant \tfrac{1}{10}a(Y)\Big\} \geqslant 1 - 2(1-r)^q$$

because $P(\Omega') \geqslant \frac{1}{5}a(Y)$ by Lemma 4. Hence, using (d4) and then Lemma 1, we arrive at the desired result.

PROOF OF THE THEOREM. We first consider the case when $\mathcal{X}$ is a field of subsets of X. Suppose $m(\mathcal{X})$ is not bounded, i.e., $c(X) > 0$. Fix any sequence $0 < t_n \downarrow 0$. We construct by induction an infinite disjoint sequence (X_n) in $\mathcal{X}$ such that for every $n \in \mathbf{N}$,

$$(*) \qquad d(t_n m(X_n)) > \tfrac{1}{320} c(X)$$

and

$$c(Y_n) > \tfrac{1}{2} c(X), \quad \text{where } Y_n = X \setminus \bigcup_{i=1}^{n} X_i.$$

Suppose the X_i's have been already defined for $i \leqslant n$ $(n \geqslant 1)$. Then we first choose $0 < r < 1$ so that $(1 - 10r)c(Y_n) > \frac{1}{2}c(X)$, and next $q \in \mathbf{N}$ satisfying $1 - 2(1 - r)^q \geqslant 0.2$. To this q, $Y = Y_n$ and $t = t_{n+1}$ we select $\mathcal{Q} \in \mathcal{P}(Y_n)$ accordingly with Lemma 4. Since $0.9 + 0.2 > 1$, by Lemma 3 with $\beta = 10$ and Lemma 5 we find $\varepsilon \in \{0, 1\}^{\mathcal{Q}}$ such that

$$c(Y_n \setminus A_\varepsilon) \geqslant (1 - 10r)c(Y_n) > \tfrac{1}{2}c(X)$$

and

$$\sup_{B \in A_\varepsilon \cap \mathcal{X}} d(t_{n+1} m(B)) \geqslant \tfrac{1}{40} a(Y_n) \geqslant \tfrac{1}{160} c(Y_n) > \tfrac{1}{320} c(X).$$

We may, therefore, choose $X_{n+1} \in A_\varepsilon \cap \mathcal{X}$ so that $(*)$ is satisfied for n replaced by $n + 1$. Since $Y_{n+1} = Y_n \setminus X_{n+1} \supset Y_n \setminus A_\varepsilon$, we have $c(Y_{n+1}) \geqslant c(Y_n \setminus A_\varepsilon) > \frac{1}{2}c(X)$. (For $n = 1$ we choose X_1 in a similar way, by applying the above procedure with $Y_0 = X$.)

Since $t_n \to 0$, condition $(*)$ means that the sequence $(m(X_n))$ is not bounded in $L_0(Z)$, contrary to the assumption of the Theorem.

If $\mathcal{X}$ is a ring of sets, then the preceding part of the proof shows that $m(A \cap \mathcal{X})$ is bounded for every $A \in \mathcal{X}$. Now boundedness of $m(\mathcal{X})$ will follow from our next lemma.

LEMMA 6. *Let L be a topological vector space and let m: $\mathcal{X} \to L$ be a finitely additive vector measure such that*:

(a) *$m(A \cap \mathcal{X})$ is bounded for every $A \in \mathcal{X}$*;

(b) *the sequence $(m(A_n))$ is bounded for every disjoint sequence (A_n) in $\mathcal{X}$.*

Then $m(\mathcal{X})$ is bounded.

PROOF. Suppose $m(\mathcal{X})$ is not bounded. Then there is a neighborhood U of 0 in L such that U does not absorb $m(\mathcal{X})$. Choose a balanced neighborhood V of 0 in L so that $V + V \subset U$. Take any $A_1 \in \mathcal{X}$ with $m(A_1) \notin V$ and next, using (a), choose $0 < t_2 < 1 = t_1$ so that $t_2 m(A_1 \cap \mathcal{X}) \subset V$. Then let $B \in \mathcal{X}$ be such that $t_2 m(B) \notin U$. Since $m(B) = m(A_1 \cap B) + m(B \setminus A_1)$ and $t_2 m(A_1 \cap B) \in V$, denoting $A_2 = B \setminus A_1$ we must have $t_2 m(A_2) \notin V$. In the next step take any $0 < t_3 < \min(\frac{1}{2}, t_2)$ so that $t_3 m((A_1 \cup A_2) \cap \mathcal{X}) \subset V$, and then choose $B \in \mathcal{X}$ with $t_3 m(B) \notin U$. Then, as before, we find that if $A_3 = B \setminus (A_1 \cup A_2)$, then $t_3 m(A_3) \notin V$.

Continuing this process we define an infinite disjoint sequence (A_n) in $\mathcal{X}$ and a sequence $0 < t_n \downarrow 0$ such that $t_n m(A_n) \notin V$ for all $n \in \mathbf{N}$, contradicting (b).

REMARKS. (1) The Theorem, and hence the Corollary as well, extends in a quite easy way to the case where Z is a metrizable locally pseudo-convex (in particular locally convex or locally bounded) topological vector space. (We follow **[3]** as concerns terminology.)

The topology of such a space Z can be determined by a sequence of F-seminorms $(\|\cdot\|_n)$ such that each $\|\cdot\|_n$ is p_n-homogeneous for some $0 < p_n \leqslant 1$ (i.e., $\|tz\|_n = |t|^{p_n}\|z\|_n$), see **[3**, p. 109 or **5**, III.2.1]. The corresponding topology of convergence in P-measure in $L_0(Z)$ is then defined by the sequence of F-seminorms (d_n), where $d_n(f) = d(\|f\|_n)$. (Measurability of Z-valued functions is understood as in the normed case.)

Now let m: $\mathcal{X} \to L_0(Z)$ be as in the Theorem. Then m is bounded if we can prove that m: $\mathcal{X} \to (L_0(Z), d_n)$ is bounded for each n. In order to see that it is really the case, fix any n and let $\|\cdot\| = \|\cdot\|_n$, $d = d_n$, $p = p_n$. Also, let $a(\cdot)$ and $c(\cdot)$ be the "measures of nonboundedness" of m corresponding to the F-seminorm $\|\cdot\|$. It suffices to verify that Lemmas 1–5 still hold true in this setting.

We start by observing that Sublemmas 1(a) and 1(b) remain valid if $\max_I \cdots$ is replaced by $\frac{1}{2}\max_I \cdots$ and $\frac{1}{4}\max_I \cdots$, respectively. In consequence, Lemmas 1 and 2 hold true. Note that p-homogeneity of $\|\cdot\|$ was not needed so far. It is, however, of importance in some of the subsequent arguments. Firstly, it is needed in the proof of Sublemma 3(a) (where t/s is to be replaced by $t/s^{1/p}$). Secondly, it implies that $a(Y) = \inf_{t>0}\sup\{d(t\|m(A)\|)\colon A \in Y \cap \mathcal{X}\}$; consequently, Lemma 4 holds true, with the same proof as Lemma 4 in **[6]**. (We note, by the way, that the latter proof requires some slight corrections due to the fact that in (3), p. 449 of **[6]**, a somewhat smaller number should be used instead of a.) The remaining sublemmas and lemmas are proved as before; of course, since Lemmas 3 and 5 depend on Sublemma 3(a) and Lemma 4, respectively, they also need p-homogeneity of $\|\cdot\|$.

(2) Although this is pretty obvious, let us, nevertheless, note explicitly that the Theorem (and the Corollary) remains valid when the probability measure P is replaced by an arbitrary positive measure μ. In this case we define a function f: $\Omega \to Z$ to be measurable if $f|\Omega'$ is measurable (in the previous sense) for all $\Omega' \in \Sigma$ with $\mu(\Omega') < \infty$. The resulting space $L_0(Z) = L_0(\Omega, \Sigma, \mu; Z)$ is then equipped with the (nonmetrizable in general) vector topology of convergence in μ-measure on sets of finite μ measure.

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