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MULTI-DIMENSIONAL ANALYTIC STRUCTURE AND SHILOV BOUNDARIES

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ABSTRACT. We give a condition under which multi-dimensional analytic structure can be introduced into the maximal ideal space of a uniform algebra.

Introduction. Let A be a uniform algebra on a compact Hausdorff space X and M the maximal ideal space of A. Various authors have showed that one-dimensional or multi-dimensional analytic structure can be introduced into M provided that there exists a suitable subset G of $\mathbf{C}$, or $\mathbf{C}^n$ respectively, with "the finite fibre property". Thus, the classic theorem on the subject by E. Bishop [**5**] states

THEOREM 1. *For $f \in A$, define $f^{-1}(\lambda) = \{x \in M: f(x) = \lambda\}$. Let W be a component of $\mathbf{C} \setminus f(X)$. Suppose that there exists a subset G of W such that:*

(1) *G has positive two-dimensional Lebesgue measure.*

(2) *For each λ in G, $\#f^{-1}(\lambda)$, the cardinality of $\{f^{-1}(\lambda)\}$, is finite.*

Then, there is an integer n such that for every $\lambda \in W$, $\#f^{-1}(\lambda) \leqslant n$. Furthermore, $f^{-1}(W)$ can be given the structure of a one-dimensional complex analytic space such that each g in A is holomorphic on this space.

B. Aupetit and J. Wermer [**1**] showed that the hypothesis on G of "positive measure" can be replaced by "positive exterior logarithmic capacity", and no weaker condition will suffice.

Also generalizing Bishop's result, R. Basener [**3**] and N. Sibony [**9**] independently formulated a condition for the existence of an n-dimensional analytic structure as follows: Let $A^n = \{(f_1,\ldots,f_n) | f_1,\ldots,f_n \in A\}$, so that each $F \in A^n$ maps M to $\mathbf{C}^n$. Let $V(F) = \{x \in M: F(x) = (0,\ldots,0)\}$. The nth Shilov boundary $\partial_n A$ is defined by $\partial_n A = \text{closure}\,[\bigcup_{F \in A^n} \partial_0 A_{V(F)}]$. $\partial_0 A$ is the usual Shilov boundary.

THEOREM 2. *Fix $F \in A^n$. Let W be a component of $F(M) \setminus F(\partial_{n-1} A)$. Suppose there exists $G \subseteq W$ such that:*

(1) *$M_{2n}(G) > 0$ (M_{2n} is the Lebesgue measure in $\mathbf{C}^n$).*

(2) *For each $\lambda \in G$, $\#F^{-1}(\lambda)$ is finite.*

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Then, there exists a positive integer k such that for all $\lambda \in W$, $\# F^{-1}(\lambda)$ is at most k. Moreover, $S = (F^{-1}(W), F, W)$ is a branched analytic cover; consequently $F^{-1}(W)$ is an analytic space and for every $f \in A$, f is holomorphic on $F^{-1}(W)$.

When $n = 1$, this is Bishop's theorem.

Recently B. Aupetit [**2**] improved Theorem 2 by replacing (1) with the condition that G is not pluri-polar. (G is not pluri-polar if there is no plurisubharmonic function ϕ on $\mathbf{C}^n$ such that $G \subset \{\lambda \in \mathbf{C}^n | \phi(\lambda) = -\infty\}$.)

Aupetit's proof of the above requires that G is contained in $F(M) \setminus F(\partial_{n-1}A)$. By the definition of the nth Shilov boundary, we have $\partial_0 A \subseteq \partial_0 A_1 \subseteq \cdots \subseteq M$. Since a component of $F(M) \setminus F(\partial_{n-1}A)$ is open in $\mathbf{C}^n$ [**3**, Lemma 2], it is contained in the interior of $F(M) \setminus F(\partial_0 A)$. In this paper we formulate a condition for an n-dimensional analytic structure assuming that G is a non-pluri-polar set contained in the interior of $F(M) \setminus F(\partial_0 A)$. The hypothesis of "non-pluri-polar set" on G is then replaced by a more general "uniqueness set". (Let Ω be a region in $\mathbf{C}^n$. $G \subseteq \overline{\Omega}$ is a *uniqueness set* (for Ω) if every plurisubharmonic function on Ω that converges to $-\infty$ at every point of G is identically equal to $-\infty$ on Ω.)

Our main results are stated in Theorems 3 and 4. We make essential use of the plurisubharmonicity proof for a certain class of functions associated with a uniform algebra, which the author developed in [7] and extended in [**8**]. We give an example covered by Theorem 4 of this paper but not by the Basener-Sibony-Aupetit Theorem.

EXAMPLE. Let Δ^2 be the open unit bidisc in $\mathbf{C}^2$ and $A = A(\overline{\Delta}^2)$. Then $\partial_0 A = \{(z, w) \in \overline{\Delta}^2 \colon |z| = 1, |w| = 1\}$; $\partial_1 A = \{(z, w) \in \overline{\Delta}^2 \colon |z| = 1 \text{ or } |w| = 1\}$; and $\partial_2 A = \overline{\Delta}^2 = M$. The set $G = \{(z, w) \colon |z| < 1, |w| = 1\}$ is a uniqueness set for Δ^2 and its 4-dimensional Lebesgue measure is zero. If we take F to be the map (z, w) then G is contained in $F(M) \setminus F(\partial_0 A)$ but not in $F(M) \setminus F(\partial_{n-1}A)$.

Theorem 4 can be readily extended to the case where $\#\{F^{-1}(\lambda)\}$ is assumed to be countable on G in view of Basener's Theorem [**4**].

We introduce some definitions and notations. Fix $F \in A^n$ and denote by $\mathscr{M}$ the subset of the Cartesian product of n-copies of M consisting of the points $m = (m_1, \ldots, m_n)$ such that $F(m_1) = \cdots = F(m_n)$. Define the projection π on $\mathscr{M}$ by $\pi(m) = F(m_1)$. Let $\mathfrak{A}$ be the uniform algebra on $\mathscr{M}$ generated by the functions of the form $\theta \to g_1(\theta_1) \cdots g_n(\theta_n)$, $g_i \in A$, $i = 1, 2, \ldots, n$. The maximal ideal space of $\mathfrak{A}$ is $\mathscr{M}$.

LEMMA 1. *Let Ω be a component of the interior of $F(M) \setminus F(\partial_0 A)$ in $\mathbf{C}^n$. For each $\tau \in \mathfrak{A}$, the function ϕ, defined by*

$$\phi(\lambda) = \log\{\max|\tau(\theta)| \colon \theta \in \pi^{-1}(\lambda)\},$$

is plurisubharmonic in Ω.

PROOF. The upper semicontinuity of ϕ is proved by a standard method (see [**11**, p. 139]). We must show that if L is a complex line contained in Ω the restriction of ϕ to L is subharmonic. Let D be a disc contained in L. For some α_{jk} and γ_k in $\mathbf{C}$,

$$L = \bigcap_{k=1}^{n-1} \left\{ (\lambda_1, \ldots, \lambda_n) \in \Omega \,\middle|\, \sum_{j=1}^{n} \alpha_{jk} \lambda_j = \gamma_k \right\}.$$

Put

$$V = \left\{ m \in \mathscr{M} \,\middle|\, \sum_{j=1}^{n} \alpha_{jk} f_j(m_1) = \gamma_k,\, k = 1,\dots,n-1 \right\}.$$

By $\mathfrak{A}_V$ we mean the restriction algebra, $\mathfrak{A}_V = \{ f \in C(V) | f$ is the uniform limit of functions in $\mathfrak{A} \}$. The maximal ideal space of $\mathfrak{A}_V$ is V. Choose a polynomial P with $\phi \leqslant \operatorname{Re} P$ on bD. Then, for each ζ in bD, $\max_{\pi^{-1}(\zeta)} |\tau| \cdot |e^{-P(\zeta)}| \leqslant 1$. We must show that $\phi \leqslant \operatorname{Re} P$ on $\bar{D}$.

Fix $z \in D$. There exists $\theta \in \pi^{-1}(z)$ such that $|\tau(\theta)| = \max_{\pi^{-1}(z)} |\tau|$. The function $\theta \to \tau(\theta) \cdot e^{-P(f_1(\theta_1),\dots,f_n(\theta_1))}$, restricted to V, is in $\mathfrak{A}_V$. We shall show in Lemma 2 that $\pi^{-1}(D) \subseteq V \setminus \partial_0[\mathfrak{A}_V]$. Assuming that Lemma 2 is true, by the local maximum modulus principle of $\mathfrak{A}_V$ applied to $\pi^{-1}(D)$,

$$|\tau(\theta)| \cdot |e^{-p(f_1(\theta_1),\dots,f_n(\theta_n))}| \leqslant |\tau(\alpha)| \cdot |e^{-p(f_1(\alpha_1),\dots,f_n(\alpha_1))}|$$

for some $\alpha = (\alpha_1,\dots,\alpha_n) \in b[\pi^{-1}(D)] \subseteq \pi^{-1}(bD)$. Hence

$$|\tau(\theta)| \cdot |e^{-p(f_1(\theta_1),\dots,f_n(\theta_1))}| \leqslant 1.$$

Thus, for an arbitrary $z \in \bar{D}$,

$$\phi(z) = \log \max\{ |\tau(\theta)| : \theta \in \pi^{-1}(z) \} \leqslant \operatorname{Re} P(z).$$

This proves Lemma 1 assuming that $\pi^{-1}(D) \subseteq V \setminus \partial_0[\mathfrak{A}_V]$, which is to be proven.

Next we deduce an important consequence of Lemma 1.

COROLLARY 1.1. *Fix $g \in A$, $F \in A^n$. For each $k \in N$,*

$$\psi_{k,g}(\lambda) = \log \max\left\{ \prod_{1 \leqslant i < j \leqslant k} |g(\theta_i) - g(\theta_j)| : \theta_i, \theta_j \in F^{-1}(\lambda) \right\}$$

is plurisubharmonic on Ω.

PROOF. $\prod_{1 \leqslant i < j \leqslant k} (g(\theta_i) - g(\theta_j)) \in \mathfrak{A}$. If $\theta_1,\dots,\theta_k \in F^{-1}(\lambda)$, then $(\theta_1,\dots,\theta_k) \in \pi^{-1}(\lambda)$. The plurisubharmonicity of $\psi_{k,g}$ follows from Lemma 1.

THEOREM 3. *Fix $g \in A$ and $F \in A^n$. Let Ω be a component in the interior of $F(M) \setminus F(\partial_0 A)$ in $\mathbf{C}^n$. Suppose that there exists a subset G of Ω with the following properties:*

(i) *G is not pluri-polar.*

(ii) *For every $\lambda \in G$, $\# \{ g \circ F^{-1}(\lambda) \}$ is finite.*

Then, there exists $k \in N$ such that for each λ in Ω, $\# \{ g \circ F^{-1}(\lambda) \}$ is at most k.

PROOF. The condition (ii) implies that $G = \bigcup_{i \in N} G_i$, where $G_i = \{ \lambda \in G | g$ assumes i values on $F^{-1}(\lambda) \}$. For some $k \in N$, G_k is non-pluri-polar. Since for each $\lambda \in G_k$, g assume k values on $F^{-1}(\lambda)$,

$$\max_{\theta_i, \theta_j \in F^{-1}(\lambda)} \prod_{1 \leqslant i < j \leqslant k+1} |g(\theta_i) - g(\theta_j)| = 0$$

on G_k. Hence, $\psi_{k+1,g} \equiv -\infty$ on Ω. This implies that g assumes at most k values on $F^{-1}(\lambda)$ for each $\lambda \in \Omega$. Take k to be the largest integer such that $\Omega_k \cap \Omega \neq \varnothing$. Thus, $\Omega = \bigcup_{i=1}^{k} \Omega_i$.

We shall now prove the hypothesis used in Lemma 1.

LEMMA 2. *Use notations as in Lemma* 1.

$$\pi^{-1}(D) \subseteq V \setminus \partial_0[\mathfrak{A}_V].$$

PROOF. Let $s \in \pi^{-1}(D)$. Then $\pi(s) = F(s_1) = (f_1(s_1),\dots,f_n(s_1)) \in D \subset L$. Therefore, $s \in V$. We need to show $s \notin \partial_0[\mathfrak{A}_V]$. Put $a = \pi(s)$. Let U be an open disc contained in L and centered at a. Let bU be the boundary of U in the topology of L. For each function H in $\mathfrak{A}_V$ we shall construct a bounded analytic function γ on U satisfying $H(s) = \gamma(a)$ and $|\gamma(a)| \leqslant \max_{\pi^{-1}(bU)}|H|$. We may assume without loss of generality that H takes the form,

$$H(\theta_1,\dots,\theta_n) = \sum_{i=1}^{l} \prod_{j=1}^{n} h_{ij}(\theta_j); \qquad (\theta_1,\dots,\theta_n) \in V, l \in N, h_{ij} \in A. \tag{1}$$

Let U_j be the projection of U on the jth coordinate axis. $F^{-1}(\bar{U}) = \bigcap_{j=1}^{n}\{f_j^{-1}(\bar{U}_j)\}$ and it is A-convex. Moreover, $F^{-1}(\bar{U}) \cap \partial_0 A = \varnothing$. Let $\tilde{f}_i$ be the restriction of f_i to $F^{-1}(\bar{U})$. By the local maximum modulus principle applied to A with respect to $F^{-1}(\bar{U})$, $\partial_0 A_{F^{-1}(\bar{U})} \subseteq \tilde{f}_j^{-1}(bU_j)$. Let μ_j be a representing measure for s_j concentrated on $\partial_0 A_{F^{-1}(\bar{U})}$. For each h_{ij} in (1), define

$$\eta_{ij}(z) = \int_{\tilde{f}_j^{-1}(bU_j)} \frac{f_j - a_j}{f_j - z} h_{ij}\, d\mu_j.$$

ASSERTION. η_{ij} has the following properties:

(i) η_{ij} is bounded and analytic on U_j.

(ii) If $\xi_j \in bU_j$ and a nontangential limit $\eta_{ij}(\xi_j) = \lim_{z\to\xi_j}\eta_{ij}(z)$ exists, then

$$|\eta_{ij}(\xi_j)| \leqslant \max_{\tilde{f}_j^{-1}(\xi_j)} |h_{ij}|.$$

(iii) $\eta_{ij}(a_j) = h_{ij}(s_j)$.

The proof for the assertion is the same as that used by Seničkin in [**10**, Lemma 7]. Using the functions, η_{ij} defined above, form a bounded analytic function γ on U as

$$\gamma(z_1,\dots,z_n) = \sum_{i=1}^{l}\prod_{j=1}^{n}\eta_{ij}(z_j).$$

Choose $\xi_j \in bU_j$ so that all the nontangential limits $\eta_{ij}(\xi_j) = \lim_{z\to\xi_j}\eta_{ij}(z)$, $z \in U_j$, $1 \leqslant i \leqslant l$, exist, and $(\xi_1,\dots,\xi_n) \in bU$. Put

$$\gamma(\xi_1,\dots,\xi_n) = \lim_{(z_1,\dots,z_n)\to(\xi_1,\dots,\xi_n)} \gamma(z_1,\dots,z_n).$$

Using essentially the same proof as in [**8**, Lemma 2] we obtain

$$|\gamma(\xi_1,\dots,\xi_n)| \leqslant \max_{\pi^{-1}(bU)} |H|.$$

This is true for almost all points in bU and γ is analytic. So,

$$|\gamma(a_1,\dots,a_n)| \leqslant \max_{\pi^{-1}(bU)} |H|.$$

Hence

$$|H(s)| \leqslant \max_{\pi^{-1}(bU)} |H|.$$

Since s and H were chosen arbitrarily and $\partial_0[\mathfrak{A}_V]$ is the closure of the generalized peak points of $\mathfrak{A}_V$, this yields the desired conclusion

$$\partial_0[\mathfrak{A}_V] \cap \pi^{-1}(D) = \varnothing.$$

Theorem 3 can be extended, using the same proof, as follows.

DEFINITION. Let Ω be a region on $\mathbf{C}^n$. We say $G \subseteq \overline{\Omega}$ is a set of uniqueness for Ω if every plurisubharmonic function defined on Ω that converges to $-\infty$ at every point of G is identically equal to $-\infty$ on Ω.

COROLLARY 3.1. *Let Ω be a component of the interior of $F(M) \setminus F(\partial_0 A)$ and $\overline{\Omega}$ its closure in $F(M) \setminus F(\partial_0 A)$. Suppose there exists a subset G of $\overline{\Omega}$ satisfying*

(i) *G is a set of uniqueness for Ω.*

(ii) *For every $\lambda \in G$, $\#\{g \circ F^{-1}(\lambda)\}$ is finite.*

Then, there exists $k \in N$ such that for each λ in Ω, $\#\{g \circ F^{-1}(\lambda)\}$ is at most k.

The following lemma is a special case of Theorem 4.

LEMMA 3. *Let A, M, and F be as before. Suppose that W is a component of $F(M) \setminus F(\partial_{n-1} A)$, and suppose $g \in A$ is constant on $F^{-1}(\lambda)$ for every $\lambda \in W$. Then, $g \circ F^{-1}$ is analytic on W.*

PROOF. We show $g \circ F^{-1}$ is analytic in each variable. Let $a \in W$ and $\Delta^n(a, r)$ be an open polydisc about a; $\Delta^n(a, r) = \prod_{i=1}^n \Delta_i$, $\Delta_i = \Delta(a_i, r_i)$. Put $\Delta_i' = \{(a_1, \ldots, z_i, \ldots, a_n) | z_i \in \Delta_i\}$. Note that $F^{-1}(\Delta_i') = f_i^{-1}(\Delta_i) \cap \Gamma$, where Γ is the set of zeros of $n-1$ functions in A. For simplicity of notation denote by f the restriction of f_i to Γ. Thus $F^{-1}(\Delta_i) = f^{-1}(\Delta_i)$, and we shall show that $g \circ f^{-1}$ is analytic on Δ_i. Consider the algebra A_Γ. By the definition of $\partial_{n-1} A$, we have $\partial_0 A_\Gamma \subseteq \partial_{n-1} A$, and $\Delta_i \cap f(\partial_0 A_\Gamma) = \varnothing$ since $\Delta(a, r) \cap F(\partial_{n-1} A) = \varnothing$. Note that $\partial_0 A_{f^{-1}(\Delta_i)} \subseteq f^{-1}(b\Delta_i)$, by the local maximum modulus principle of A_Γ applied to $f^{-1}(\Delta_i)$.

Let μ_i be the representing measure for some $m_i \in f^{-1}(a_i)$ supported on $\partial_0 A_{f^{-1}(\Delta_i)}$, and ν_i the projection of μ_i on $b\Delta_i$, which is the normalized Lebesgue measure on $b\Delta_i$.

$$\int_{b\Delta_i} g \circ f^{-1} \, d\nu_i = \int_{f^{-1}(b\Delta_i)} g \, d\mu_i = g(m_i) = g \circ f^{-1}(a_i).$$

Thus, $g \circ f^{-1}$ is a complex harmonic function.

$$\int_{b\Delta_i} (z - a_i)^n \cdot g \circ f^{-1} \, d\nu_i = \int_{f^{-1}(b\Delta_i)} (f - a_i)^n \cdot g \, d\mu_i = 0 \qquad (n \in N).$$

This shows that $g \circ f^{-1}$ is holomorphic on Δ_i and hence, $g \circ F^{-1}$ on Δ_i'.

THEOREM 4. *Fix $F \in A^n$ and $g \in A$. Let Ω be a component contained in the interior of $F(M) \setminus F(\partial_0 A)$, and $\overline{\Omega}$ its closure in $F(M) \setminus F(\partial_0 A)$. Suppose there exists a subset $G \subset \overline{\Omega}$ such that*:

(i) *G is a set of uniqueness for Ω.*

(ii) *For every* $\lambda \in G$, $\#\{g \circ F^{-1}(\lambda)\}$ *is finite.*

Let W *be an open connected subset of* Ω *such that* $W \cap F(\partial_{n-1}A) = \varnothing$. *Then, there exists* $k \in N$ *such that the mapping* $F\colon F^{-1}(W) \to W$ *is a* k*-sheeted analytic covering, so that every* $f \in A$ *is holomorphic on* $F^{-1}(W)$.

PROOF OF THEOREM 4. By Corollary 3.1 there is a $k \in N$ such that $W = \bigcup_{i=1}^{k} W_i$, where $W_i = \{\lambda \in W | g$ assumes i values on $F^{-1}(\lambda)\}$. Without loss of generality assume $W_k \neq \varnothing$. We shall show $g \circ F^{-1}$ is analytic in W_k. Let $\lambda \in W_k$ and $b_1, \ldots, b_k$ be the distinct values of g on $F^{-1}(\lambda)$. Let $D_i \subset \mathbf{C}$ $(1 \leqslant i \leqslant k)$ be a disc centered at b_i. Assume $\overline{D}_i \cap \overline{D}_j = \varnothing$ for $i \neq j$.

ASSERTION 1. There exists a neighborhood N of λ such that $g(F^{-1}(\overline{N})) \subset \bigcup_{i=1}^{k} D_i$ and $\overline{N} \subseteq W$. This follows from continuity of g and the topology of M.

ASSERTION 2. Let $e_i = F^{-1}(\overline{N}) \cap g^{-1}(\overline{D}_i)$. Then, $F(e_i) = \overline{N}$.

PROOF. Denote $F|_{e_i} = F_i$. Apply Lemma 1 of [**3**] to $A_{F^{-1}(\overline{N})}$ and e_i to obtain $\partial_{n-1}A_{e_i} \subseteq F_i^{-1}(b\overline{N})$. Recall that $\lambda \in \mathbf{C}^n \setminus F(\partial_{n-1}A_{e_i})$. If J is a component with $\lambda \in J \subseteq \mathbf{C}^n \setminus F(\partial_{n-1}A_{e_i})$, by Lemma 2 of [**3**], $F(e_i) \cap J = J$. J contains N and $F(e_i)$ is closed. So, $N \subseteq \overline{N} \subseteq F(e_i)$. We have $F(e_i) \subseteq \overline{N}$ by the definition of e_i.

Since g assumes at most k-values on every fiber, it is clear that g is constant on the set $F^{-1}(\lambda) \cap e_i$ $(\lambda \in \overline{N}, 1 \leqslant i \leqslant k)$. In view of Lemma 3, $\lambda \mapsto g \circ F^{-1}(\lambda)$ is analytic on N. We have shown that $F^{-1}(W_k) \to W_k$ is a k-sheeted covering map, and also that W_k is open.

Next, using Assertion 2 above and the ideas of Bishop and Basener [**3**, p. 103], we can show that $W \setminus W_k$ is a negligible set in W and $F^{-1}(W_k)$ is dense in $F^{-1}(W)$.

Consequently, we conclude that $(F^{-1}(W), F, W)$ is a k-sheeted analytic cover in the sense of [**6**, p. 101]. This proves Theorem 4.

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