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COMPACT 3-MANIFOLDS WITH INFINITELY-GENERATED GROUPS OF SELF-HOMOTOPY-EQUIVALENCES

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ABSTRACT. Examples are constructed of compact 3-manifolds with boundary whose groups of self-homotopy-equivalences are not finitely-generated.

For a finite CW-complex X, let $\mathcal{E}(X)$ denote the space of basepoint-preserving homotopy equivalences from X to X, and let $\mathcal{G}(X)$ denote the *group of homotopy equivalences* $\pi_0(\mathcal{E}(X))$. An obvious question is: Under what conditions on X must $\mathcal{G}(X)$ be finitely-generated? Sullivan [7] and Wilkerson [**10**] showed that if X is simply-connected, then $\mathcal{G}(X)$ is finitely-presented. For non-simply-connected complexes, however, $\mathcal{G}(X)$ can be infinitely-generated (have no finite generating set) even for seemingly uncomplicated examples. Frank and Kahn [**3**] showed that $\mathcal{G}(S^1 \vee S^p \vee S^{2p-1})$ is infinitely-generated when $p \geqslant 2$, and in [**5**] the author gave infinitely many examples of finite four-dimensional $K(\pi, 1)$-complexes with $\mathrm{Aut}(\pi)$ and, hence, $\mathcal{G}(K(\pi, 1))$ infinitely-generated. In [**5**], it was asked whether there was an example of an aspherical 2-complex X with $\mathcal{G}(X)$ infinitely-generated. Recently, such examples were found by Brunner and Ratcliffe [**2**].

These various examples show two distinct ways that $\mathcal{G}(X)$ can fail to be finitely-generated. In the Frank and Kahn examples, $\pi_{2p-1}(S^1 \vee S^p \vee S^{2p-1})$ is quite large—it is infinitely-generated as a $\mathbf{Z}\pi_1(S^1 \vee S^p \vee S^{2p-1})$-module—and many elements of $\mathcal{G}(S^1 \vee S^p \vee S^{2p-1})$ arise by mapping the S^{2p-1} using an element of $\pi_{2p-1}(S^1 \vee S^p \vee S^{2p-1})$. In the aspherical examples, $\mathrm{Aut}(\pi_1(X))$ is infinitely-generated and the asphericity forces $\mathcal{G}(X) \to \mathrm{Aut}(\pi_1(X))$ to be surjective. Clearly, if $\mathcal{G}(X) \to \mathrm{Aut}(\pi_1(X))$ is surjective then so is

$$\mathcal{G}(X \vee S^n) \to \mathrm{Aut}(\pi_1(X \vee S^n)) \quad \text{for } n \geqslant 2,$$

so one can produce nonaspherical examples in dimension two. What is not apparent from these examples is the answer to the following question posed in [**2**].

Question. Is there a finite two-dimensional complex X with $\mathrm{Aut}(\pi_1(X))$ finitely-generated but $\mathcal{G}(X)$ not finitely-generated?

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For a 3-manifold M let M' denote the result of removing from M the interiors of two disjoint closed 3-balls tamely-imbedded in the interior of M. Our main result is

THEOREM 1. *Let M be a compact aspherical 3-manifold-with-boundary, such that* $\mathrm{Out}(\pi_1(M))$ *is finite and* $\pi_1(M)$ *admits a surjective homomorphism onto* $\mathbf{Z} \times \mathbf{Z}$. *Then* $\mathcal{G}(M')$ *is infinitely-generated.*

Many instances of Theorem 1 are given in

COROLLARY. *Let M be a compact orientable 3-manifold-with-boundary such that the interior of M admits a complete hyperbolic structure with finite volume, and such that* $\pi_1(M)$ *admits a surjective homomorphism onto* $\mathbf{Z} \times \mathbf{Z}$. *Then* $\mathcal{G}(M')$ *is infinitely-generated.*

PROOF. M is aspherical since its interior is, and $\mathrm{Out}(\pi_1(M))$ is well known to be finite [**6**, p. 116; **8**, p. 5.31]. □

For example, the (compact) complement of the Whitehead link and the (compact) complement of the Borromean rings are familiar 3-manifolds which satisfy the hypotheses of the theorem and the corollary.

Since any compact 3-manifold-with-boundary has the homotopy type of a finite 2-complex, and $\mathrm{Out}(\pi_1(M'))$ finitely-generated implies $\mathrm{Aut}(\pi_1(M'))$ finitely-generated, Theorem 1 answers the question of Brunner and Ratcliffe in the affirmative.

We will give the proof of Theorem 1 in §1, making use of two auxiliary theorems. These theorems, which are of independent interest, are proved in §§2 and 3. I wish to thank Andy Miller for helpful discussions concerning Theorem 2(b).

1. Proof of Theorem 1. Write π for $\pi_1(M, *) \cong \pi_1(M', *)$. Let $\Phi\colon \mathcal{G}(M') \to \mathrm{Aut}(\pi)$ be the homomorphism defined by $\Phi(\langle f \rangle) = f_\#$. Let $\mathcal{G}_1(M') = \Phi^{-1}(\{1\})$ and $\mathcal{G}_{\mathrm{Inn}}(M') = \Phi^{-1}(\mathrm{Inn}(\pi))$. Since $\mathrm{Out}(\pi)$ is finite, $\mathcal{G}_{\mathrm{Inn}}(M')$ has finite index in $\mathcal{G}(M')$, so to prove the theorem it suffices to show $\mathcal{G}_{\mathrm{Inn}}(M')$ is infinitely-generated.

Let $M_1 = M$ if M is orientable, otherwise let M_1 be the orientable double cover of M. Now M_1 is compact, orientable, and has a boundary component which is not a 2-sphere. Therefore, $H_1(M_1; \mathbf{Z})$ is infinite so M_1 is sufficiently large. Therefore, the center of $\pi_1(M_1)$ is finitely-generated [**9**]. This implies that the center of π is finitely-generated. Using Theorem 2(b), which will be stated and proved in §2, we see that $\mathcal{G}_{\mathrm{Inn}}(M')$ is infinitely-generated if $\mathcal{G}_1(M')$ is.

Let $\mathrm{Aut}_\pi(\pi_2(M'))$ be the group of π-module automorphisms of $\pi_2(M')$. We will prove that the natural homomorphism $\mathcal{G}_1(M') \to \mathrm{Aut}_\pi(\pi_2(M'))$ is surjective. Let K be a finite 2-complex having the homotopy type of M; then K is aspherical and $K' = K \vee S^2 \vee S^2$ has the homotopy type of M'. Since π has cohomological dimension two, the k-invariant $k(K')$ is zero. As shown in [**2**], this implies $\mathcal{G}_1(K') \to \mathrm{Aut}_\pi(\pi_2(K'))$ is surjective. (This surjectivity can be proved directly for K' without difficulty: just define a homotopy equivalence that is the identity on K and induces the desired automorphism on $\pi_2(K')$.) Therefore, $\mathcal{G}_1(M') \to \mathrm{Aut}_\pi(\pi_2(M'))$ is surjective, so $\mathcal{G}_1(M')$ is infinitely-generated if $\mathrm{Aut}_\pi(\pi_2(M'))$ is. But $\pi_2(M') \cong \pi_2(K') \cong \mathbf{Z}\pi \oplus \mathbf{Z}\pi$, so $\mathrm{Aut}_\pi(\pi_2(M')) \cong \mathrm{GL}_2(\mathbf{Z}\pi)$, the group of 2×2 invertible matrices with entries in $\mathbf{Z}\pi$. We apply Theorem 3, which will be stated and proved in §3, with $G = \pi$ to show that $\mathrm{GL}_2(\mathbf{Z}\pi)$ is infinitely-generated. This completes the proof of Theorem 1. □

2. Proof of Theorem 2. While part (a) of Theorem 2 is not needed in the proof of Theorem 1, it is of independent interest and can be proved without much more work than that needed for part (b). A special case of part (a) appears in [**4**].

THEOREM 2. *Let L be a finite-dimensional locally-finite connected simplicial complex. Then*:

(a) *If $\pi_1(L,*)$ is centerless, then $\mathscr{G}_{\mathrm{Inn}}(L) \cong \mathscr{G}_1(L) \times \pi_1(L,*)$.*

(b) *If the center of $\pi_1(L,*)$ is finitely-generated, then $\mathscr{G}_{\mathrm{Inn}}(L)$ is infinitely-generated if and only if $\mathscr{G}_1(L)$ is infinitely-generated.*

PROOF. Write π for $\pi_1(L,*)$. Replacing L by the interior of a regular neighborhood of L, we may assume L is a triangulated open manifold, and the basepoint $*$ is a vertex of L. Let N be a regular neighborhood in L of the 1-skeleton of L. Define α: $\pi \to \mathscr{G}_{\mathrm{Inn}}(L)$ as follows. For each $\sigma \in \pi$, choose an isotopy H_σ: $L \times I \to L$ starting at the identity map 1_L so that the trace of H_σ (the homotopy class of the restriction of H_σ to $* \times I$) equals σ^{-1}, and so that the restriction of H_σ to $(L - \mathrm{int}(N)) \times \{t\}$ equals the identity for all $t \in I$. Let $h_\sigma(x) = H_\sigma(x,1)$. Note that for $\tau \in \pi$, $(h_\sigma)_\#(\tau) = \sigma\tau\sigma^{-1}$, so we can define $\alpha(\sigma) = \langle h_\sigma \rangle$.

We will now show that α is a homomorphism. For homotopies G, H: $L \times I \to L$ with $G(x,1) = H(x,0)$, we define $(G * H)(x,t)$ to be $G(x,2t)$ if $0 \leqslant t \leqslant \frac{1}{2}$ and to be $H(x,2t-1)$ if $\frac{1}{2} \leqslant t \leqslant 1$. We define $\overline{G}(x,t)$ to be $G(x,1-t)$. Suppose σ, $\tau \in \pi$. Then $H_\sigma * (h_\sigma \circ H_\tau)$ is a homotopy from 1_L to $h_\sigma h_\tau$ with trace $\sigma^{-1} \cdot \sigma\tau^{-1}\sigma^{-1} = (\sigma\tau)^{-1}$. Therefore the trace of $\overline{H_{\sigma\tau}} * (H_\sigma * (h_\sigma \circ H_\tau))$ is 1 so $\langle h_{\sigma\tau} \rangle = \langle h_\sigma h_\tau \rangle = \langle h_\sigma \rangle \langle h_\tau \rangle$.

Next, we will show that if $\langle f \rangle \in \mathscr{G}_1(L)$ and $\langle h_\sigma \rangle \in \mathrm{im}(\alpha)$, then $\langle f \rangle \langle h_\sigma \rangle = \langle h_\sigma \rangle \langle f \rangle$. We may choose f within its homotopy class so that $f|_N = 1_N$. A homotopy G: $fh_\sigma \simeq h_\sigma f$ is defined by $G(x,t) = h_\sigma(x)$ if $x \in N$ and $G(x,t) = H_\sigma(f(x),t)$ if $x \in L - \mathrm{int}(N)$.

Let μ: $\pi \to \mathrm{Inn}(\pi)$ send σ to the inner automorphism $\mu(\sigma)(\tau) = \sigma\tau\sigma^{-1}$. When π is centerless, μ is an isomorphism and $\alpha \circ \mu^{-1}$ provides a splitting in the exact sequence $1 \to \mathscr{G}_1(L) \to \mathscr{G}_{\mathrm{Inn}}(L) \to \mathrm{Inn}(\pi) \to 1$. Since $\mathrm{im}(\alpha)$ commutes with $\mathscr{G}_1(L)$, this establishes part (a). For (b), observe that $\mathscr{G}_1(L) \cap \mathrm{im}(\alpha)$ is the image under α of the center of π, so it will be finitely-generated when the center of π is. But inclusion induces an isomorphism $\mathscr{G}_1(L)/(\mathscr{G}_1(L) \cap \mathrm{im}(\alpha)) \cong \mathscr{G}_{\mathrm{Inn}}(L)/\mathrm{im}(\alpha)$, and (b) follows. □

3. Proof of Theorem 3. Theorem 3 is proved by a modification of an argument of Bachmuth and Mochizuki.

THEOREM 3. *Let G be any group admitting a surjective homomorphism η: $G \to \mathbf{Z} \times \mathbf{Z}$. Then $\mathrm{GL}_2(\mathbf{Z}G)$ is infinitely-generated.*

PROOF. Let s and t be generators of $\mathbf{Z} \times \mathbf{Z}$, and denote the group ring $\mathbf{Z}[\mathbf{Z} \times \mathbf{Z}] = \mathbf{Z}[s, s^{-1}, t, t^{-1}]$ by R. The homomorphism η induces a homomorphism β: $\mathrm{GL}_2(\mathbf{Z}G) \to \mathrm{GL}_2(R)$. Let $S = \mathrm{SL}_2(R) \cap \mathrm{im}(\beta)$.

We begin by using an idea from [**2**] to show that $\mathrm{GL}_2(\mathbf{Z}G)$ is infinitely-generated if S is. There is a short exact sequence

$$1 \to \mathrm{SL}_2(R) \to \mathrm{GL}_2(R) \overset{\det}{\to} R^* \to 1$$

where the group of units R^* is generated by $\{-1, s, t\}$. Let $R_0 \subset R^*$ be the subgroup generated by $\{s^2, t^2\}$, which has index 8 in R^*, and let $H = \det^{-1}(R_0)$. If $w \in R_0$, then the "positive" square root $w^{1/2}$ is uniquely defined, and f: $H \to \mathrm{SL}_2(R)$ defined by $f(A) = (\det(A))^{-1/2}A$ is a retraction. Let $K = \mathrm{image}(\beta)$. Since H has finite index in $\mathrm{GL}_2(R)$, $K \cap H$ has finite index in K. But $f|_{K \cap H}$ retracts $K \cap H$ onto S. Therefore, if S is infinitely-generated, then so is $\mathrm{GL}_2(\mathbf{Z}G)$.

The proof that S is infinitely-generated is a minor modification (for the case $P = \mathbf{Z}$) of the argument of §2 of [**1**], and we use the notation of that paper. Choose $x, y \in G$ with $\eta(x) = s$ and $\eta(y) = t$. Lemma 1 of [**1**] is replaced by

LEMMA 1′. *$E_2(R)$ is contained in*

$$\left(S \cap \mathrm{SL}_2(\mathbf{Z}[s, s^{-1}, t])\right) *_V \left(S \cap \mathrm{SL}_2(\mathbf{Z}[s, s^{-1}, t])^{\left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right]}\right),$$

where V is the intersection of the factors.

LEMMA 2 is replaced by

LEMMA 2′. *Let π be a nonunit element of $\mathbf{Z}$. Then, the matrices*

$$\begin{bmatrix} 1 & 0 \\ (s-1)/\pi^i & 1 \end{bmatrix}, \qquad i \geqslant 1,$$

can be chosen as part of a set of double coset representatives of $(S \cap \mathrm{SL}_2(\mathbf{Z}[s, s^{-1}, t]), U)$ in $\mathrm{SL}_2(\mathbf{Q}[s, s^{-1}, T])$.

The proof is unchanged. Alternatively, since

$$S \cap \mathrm{SL}_2(\mathbf{Z}[s, s^{-1}, t]) \subseteq \mathrm{SL}_2(\mathbf{Z}[s, s^{-1}, t]),$$

Lemma 2 implies Lemma 2′.

The argument continues exactly as in [**1**]. In the final calculation, one must check that

$$M_i \begin{bmatrix} 1 & \pi^{2i}t^{-1} \\ 0 & 1 \end{bmatrix} M_i^{-1} = \begin{bmatrix} 1 - \pi^i(s-1)t^{-1} & \pi^{2i}t^{-1} \\ -(s-1)^2 t^{-1} & 1 + \pi^i(s-1)t^{-1} \end{bmatrix}$$

is in S. Let D_i be this matrix, and let D_i' be the matrix

$$D_i' = \begin{bmatrix} 1 - \pi^i y^{-1}(x-1) & \pi^{2i}y^{-1} \\ -(x-1)y^{-1}(x-1) & 1 + \pi^i(x-1)y^{-1} \end{bmatrix}$$

with entries in $\mathbf{Z}G$. Then D_i' is invertible with two-sided inverse

$$\begin{bmatrix} 1+\pi^i y^{-1}(x-1) & -\pi^{2i}y^{-1} \\ (x-1)y^{-1}(x-1) & 1-\pi^i(x-1)y^{-1} \end{bmatrix}$$

and $\beta(D_i') = D_i$. $\square$

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