PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 1, September 1984

COEFFICIENT BOUNDS FOR THE INVERSE OF A FUNCTION WITH DERIVATIVE IN $\mathcal{P}$. II

RICHARD J. LIBERA AND ELIGIUSZ J. ZŁOTKIEWICZ

ABSTRACT. Sharp upper bounds are given for the coefficients of the inverse of a normalized univalent function whose derivative has positive real part. This resolves a question raised by these authors in Proc. Amer. Math. Soc. **87** (1983), 251–257.

The family of all functions of the form

$$f(z) = z + a_2 z^2 + \cdots, \tag{1}$$

regular and one-to-one in the open unit disk Δ, $\Delta = \{z \in \mathbf{C}\colon |z| < 1\}$, is (as usual) denoted by S. The class S has been the subject of much inquiry (see Bernardi [**1**] for example).

The inverse of $f(z)$ has a series expansion in a disk centered at the origin with the form

$$\check{f}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots. \tag{2}$$

The search for bounds on the magnitude of the coefficients γ_k has raised some interesting and unexpected questions (see [**4**–**8**] or [**3**, §7, Chapter 16]).

Let $\mathcal{P}$ be the family of all functions

$$P(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{3}$$

regular in Δ and satisfying $\operatorname{Re}\{P(z)\} > 0$ for z in Δ. The normalized integral of any function in $\mathcal{P}$ is a member of S [**3**], i.e.,

$$f(z) = \int_0^z P(\varsigma)\, d\varsigma \tag{4}$$

is in S if $P(z)$ is in $\mathcal{P}$; we denote all functions formed in this way by $\mathcal{I}$.

Earlier [**7**] the authors showed that if $f(z)$ is in $\mathcal{I}$, then the coefficients of $\check{f}(w)$, given in (2), are restricted so that

$$|\gamma_2| \le 1, \quad |\gamma_3| \le \frac{4}{3}, \quad |\gamma_4| \le \frac{13}{6}, \quad |\gamma_5| \le \frac{59}{15} \quad \text{and} \quad |\gamma_6| \le \frac{344}{45}. \tag{5}$$

It was shown that these bounds are rendered sharp only for the inverse of the function

$$f_0(z) = -z - 2\log(1-z), \tag{6}$$

which corresponds, as in (4), to the member $P_0(z)$ of $\mathcal{P}$ given by

$$P_0(z) = \frac{1+z}{1-z} = 1 + 2\sum_{k=1}^{\infty} z^k, \quad \text{for } z \text{ in } \Delta. \tag{7}$$

Received by the editors September 15, 1983.
1980 *Mathematics Subject Classification*. Primary 30C45, 30C50.

Furthermore, the authors suggested that it is quite likely that $\check{f}_0(w)$, the inverse of (6), gives sharp upper bounds for other (perhaps all) coefficients exhibited in (2). It is the purpose of this note to show that the authors' suggestion is correct.

Specifically, the authors propose the following

THEOREM. *If $f(z)$ is any member of $\mathcal{I}$,*

$$\check{f}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots$$

and

$$\check{f}_0(w) = w + B_2 z^2 + B_3 z^3 + \cdots$$

is the inverse of $f_0(z)$ given in (6), then

$$|\gamma_k| \le |B_k|, \qquad k = 2, 3, 4, \ldots, \tag{8}$$

and $\check{f}_0(w)$ is the unique function giving equality in (8).

PROOF. If $P(z)$ is in $\mathcal{P}$ then the reciprocal $1/P(z)$ is likewise in $\mathcal{P}$, and conversely. Consequently (4) can be written as $f'(z) = 1/P(z)$ and this, combined with the relation $f(\check{f}(w)) = w$, gives the equation

$$\check{f}'(w) = P(\check{f}(w)), \qquad |w| < \rho(f). \tag{9}$$

Using representations (2) and (3) enables us to rewrite (9) as

$$1 + \sum_{k=2}^{\infty} k\gamma_k w^{k-1} = 1 + \sum_{k=1}^{\infty} C_k(\check{f}(w))^k, \tag{10}$$

from which, after some work, one obtains

$$\begin{aligned} 2! \cdot \gamma_2 &= C_1, \qquad 3!\gamma_3 = C_1^2 + 2C_2, \\ 4! \cdot \gamma_4 &= C_1^3 + 8C_1C_2 + 6C_3, \\ 5! \cdot \gamma_5 &= C_1^4 + 22C_1^2C_2 + 42C_1C_3 + 16C_2^2 + 24C_4, \\ 6! \cdot \gamma_6 &= C_1^5 + 52C_1^3C_2 + 192C_1^2C_3 + 136C_1C_2^2 \\ &\quad + 264C_1C_4 + 180C_2C_3 + 120C_5. \end{aligned} \tag{11}$$

Furthermore, a careful examination of the formation of the coefficients in (11), and those with higher index, reveals that

$$k! \cdot \gamma_k = C_1^{k-1} + R_k(C_1, C_2, \ldots, C_{k-1}) \tag{12}$$

for each k, R_k being a polynomial whose coefficients are all nonnegative.

As a consequence, a sharp upper bound for (12) may be obtained by a direct application of the triangle inequalitty along with the bounds $|C_k| \le 2$ for all k [**3**]. The (unique) function extremizing $|C_1|$ and all subsequent $|C_k|$ is $P_0(z)$, given in (7). This concludes the proof.

The function $\check{f}_0(w)$ of the theorem has the Maclaurin series

$$\check{f}_0(w) = w - w_2 + \frac{4}{3}w^3 - \frac{13}{6}w^4 + \frac{59}{15}w^5 - \frac{344}{45}w^6 + \cdots + B_k w^k + \cdots, \tag{13}$$

and the coefficients are defined recursively by

$$(k+1)B_{k+1} + B_k + \sum_{j=1}^{k}(k+1-j)B_jB_{k+1-j} = 0, \tag{14}$$

for $k = 1, 2, \ldots$ and $B_1 = 1$.

In comparison with the authors' earlier attempt [**7**] of a proof of this theorem, the above is straightforward and simple. This is largely due to the observation that replacing $P(z)$ in the equation $f'(z) = P(z)$ by its reciprocal gives representations for the γ_k's (in terms of C_k's as in (11)) in a far more tractable form. This observation was made by Jan Campschroer in a related work [**2**].

Suppose $f(z)$ is an odd function in $\mathcal{I}$. Then its inverse is likewise odd and, in this case, the corresponding function $P(z)$ defined in (4) must be even. An application of the preceding technique justifies the following statement.

COROLLARY. *If $f(z) = z + a_3z^3 + a_5z^5 + \cdots$ is an odd member of $\mathcal{I}$, then the coefficients of its inverse*

$$\check{f}(w) = w + \beta_3 w^3 + \beta_5 w^5 + \cdots, \tag{15}$$

satisfy the (sharp) inequalities

$$|\beta_{2k+1}| \le |C_{2k+1}|, \qquad k = 1, 2, \ldots; \tag{16}$$

the numbers C_{2k+1} are the coefficients of the inverse of $g_0(z)$ in $\mathcal{I}$, which is obtained by letting $P(z)$ be $P_0(z^2)$ in (4).

The function defined in the corollary is

$$g_0(z) = -z + \log\left(\frac{1+z}{1-z}\right)$$

and its inverse is

$$\check{g}_0(w) = w - \frac{2}{3}w^3 + \frac{14}{15}w^5 - \frac{538}{315}w^7 + \frac{13,478}{2,835}w^9 + \cdots. \tag{17}$$

The functions in $\mathcal{I}$ which are k-fold symmetric can be handled by the method of the corollary.

[This work was done at Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin, Poland, under an exchange program between the United States National Academy of Sciences and Polska Akademia Nauk.]

REFERENCES

1. S. D. Bernardi, *Bibliography of schlicht functions*, Mariner, Tampa, Fla., 1982.
2. J. T. P. Campschroer, *Coefficients of the inverse of a convex function*, Dept. of Math. Rep. No. 8227, November 1982, Catholic Univ., 6525 ED Nijmegen, The Netherlands.
3. A. W. Goodman, *Univalent functions*, Vols. 1, 2, Mariner, Tampa, Fla., 1983.
4. W. E. Kirwan and G. Schober, *Inverse coefficients for functions of bounded boundary rotation*, J. Analyse Math. **36** (1979), 167–178.
5. J. G. Krzyż, R. J. Libera and E. J. Złotkiewicz, *Coefficients of inverses of regular starlike functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **33** (1979), 103–110 (1981).
6. R. J. Libera and E. J. Złotkiewicz, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc. **85** (1982), 225–230.
7. ——, *Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$*, Proc. Amer. Math. Soc. **87** (1983), 251–257.
8. G. Schober, *Coefficient estimates for inverses of schlicht functions*, Aspects of Contemporary Complex Analysis, Academic Press, New York, 1980, pp. 503–513.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DELAWARE, NEWARK, DELAWARE 19711

INSTYTUT MATEMATYKI, UMCS, UL. NOWOTKI 10, 20-031 LUBLIN, POLAND