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L^p-COMPUTABILITY IN RECURSIVE ANALYSIS

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ABSTRACT. L^p-computability is defined in terms of effective approximation; e.g. a function $f \in L^p[0,1]$ is called L^p-computable if f is the effective limit in L^p-norm of a computable sequence of polynomials. Other families of functions can replace the polynomials; see below. In this paper we investigate conditions which are not based on approximation. For $p > 1$, we show that f is L^p-computable if and only if (a) the sequence of Fourier coefficients of f is computable, and (b) the L^p-norm of f is a computable real. We show that this fails for $p = 1$.

In [**9**] the authors gave a definition of L^p-computability for functions on $[0,1]$ (where $1 \leq p < \infty$ and p is a computable real). Various equivalent formulations were given, based on Weierstrass approximation (polynomials), Fourier series (trigonometric polynomials) and integration theory (step functions). Thus a function $f \in L^p[0,1]$ is called *L^p-computable* if f is the effective limit in L^p-norm of any of the following:

(i) a computable sequence of polynomials;

(ii) a computable sequence of trigonometric polynomials;

(iii) a computable sequence of step functions (i.e. a sequence in which the jump points and heights of the steps are computable).

Each of these definitions is based on a process of generation of the function f from simpler functions. From the point of view of an analyst, it is useful to have a definition based on the properties of the function f itself. In this paper we prove:

For $1 < p < \infty$, a function $f \in L^p[0,1]$ is L^p-computable if and only if:

(a) *the sequence of Fourier coefficients of f is computable, and*

(b) *the L^p-norm of f is computable.*

L^p-computability is a generalization of the classical Grzegorczyk notion of computability for a continuous function (Grzegorczyk [**5, 6**], Lacombe [**7**]). Grzegorczyk computability can be formulated as follows: a continuous function f on $[0,1]$ is *computable* (in the sense G) if f is the effective limit in the uniform norm of a computable sequence of polynomials (Pour-El and Caldwell [**8**]). We note the similarity to (i)–(iii) above.

We mention in passing that conditions (i)–(iii) can now be expanded to include:

(iv) a function $f \in L^p[0,1]$ is *L^p-computable* if f is the effective limit in L^p-norm of a G-computable sequence of continuous functions.

[A more general notion, encompassing (i)–(iv), is that of an *effective generating set*; cf. [**9**].]

However, as noted above, the criterion given in this paper (for $1 < p < \infty$) is of a different type. One might ask why this criterion fails for $p = 1$. One reason, loosely

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speaking, is that in the case of L^1 the norm gives very little information in addition to that given by the Fourier coefficients: e.g. if $f \geq 0$, then the L^1-norm of f is nothing but the zeroth Fourier coefficient. A more precise reason is the property of "uniform convexity", which holds for L^p, $1 < p < \infty$, but fails for $p = 1$. This property (see the Clarkson inequalities below) is used in our proof.

Condition (a) (computability of the sequence of Fourier coefficients), when taken by itself, is called *weak L^p-computability.* It is shown in [**9**] that weak computability does not imply (strong) L^p-computability. Thus the extra hypothesis (b) (computability of the norm) is necessary for our result. However, weak computability is of some independent interest, and it is useful to observe the following:

For $1 < p < \infty$, a function $f \in L^p[0,1]$ is *weakly L^p-computable* if and only if it maps every (strongly) L^q-computable sequence $g_n \in L^q[0,1]$ (with $p^{-1} + q^{-1} = 1$) onto a computable sequence of real or complex numbers $\int_0^1 f(x) g_n(x)\, dx$. (Here, of course, we are considering the linear functional on L^q associated with f by virtue of the Riesz representation theorem.) To prove this we simply observe that the computable sequence g_n is the effective limit in L^q-norm of a computable double sequence of trigonometric polynomials.

REMARKS. There is an easier treatment of this topic in the case $p = 2$. Let $\|f\|$ denote the L^2-norm of f, and $\{c_n\}$ its sequence of Fourier coefficients. Then $\|f\|^2 = \sum |c_n|^2$. Suppose we have (a) that the sequence $\{c_n\}$ is computable. Then the series $\sum |c_n|^2$ converges effectively if and only if its limit $\|f\|^2$ is a computable real (condition (b)). Unfortunately this proof does not work for $p \neq 2$.

For additional recent work on the connection between logic and analysis, see e.g. Aberth [**1**], Feferman [**3**], Friedman [**4**] and Simpson [**10**].

We come now to the formal statement of our main results and their proofs.

THEOREM 1. *Let p be a computable real, $1 < p < \infty$. Then a function $f \in L^p[0,1]$ is (strongly) L^p-computable if and only if*

(a) *the sequence of Fourier coefficients is computable,*

(b) *the L^p-norm of f is computable.*

THEOREM 2. *The result in Theorem 1 fails when $p = 1$.*

PROOF OF THEOREM 1. This result depends on the "uniform convexity" of the L^p-spaces for $p > 1$. We use the following inequalities due to Clarkson [**2**, p. 400]. Here $p^{-1} + q^{-1} = 1$ and the norm in every case is the p-norm.

$$(1) \qquad \begin{aligned} &\|f+g\|^p + \|f-g\|^p \leq 2^{p-1}(\|f\|^p + \|g\|^p) && \text{for } p \geq 2, \\ &\|f+g\|^q + \|f-g\|^q \leq 2(\|f\|^p + \|g\|^p)^{q-1} && \text{for } 1 < p \leq 2. \end{aligned}$$

As a preliminary step, we pass to "dyadic step functions", i.e. step functions whose jump discontinuities occur at dyadic rationals $w/2^t$, $0 \leq w \leq 2^t$. A dyadic step function is said to belong to the "tth generation" if t is the largest exponent of 2 appearing in its definition.

LEMMA 1. *A function $f \in L^p[0,1]$, $1 \leq p < \infty$ is (strongly) L^p-computable if and only if there is a computable sequence of dyadic step functions which converges effectively to f in L^p-norm.*

PROOF. This is clear, since the computable continuous functions can be effectively approximated in L^p-norm by dyadic step functions, and conversely.

LEMMA 2. *A function $f \in L^p[0,1]$, $1 < p < \infty$, is weakly L^p-computable if and only if it maps the computable sequence of characteristic functions χ_i of dyadic intervals $(w/2^t,\ (w+1)/2^t]$ onto a computable sequence of real or complex numbers. For $p = 1$ this condition is sufficient for weak L^p-computability.*

PROOF. For $p > 1$, we use the fact that computable trigonometric polynomials can be effectively approximated in L^q-norm by computable dyadic step functions, and conversely. For $p = 1$ (so that $q = \infty$), computable trigonometric polynomials can still be effectively approximated in the uniform norm by computable dyadic step functions (although not conversely). Q.E.D.

Now we come to the main part of the proof. One half of Theorem 1, that strong computability implies weak computability and computability of the norm, is trivial. For the converse, assume that f is weakly computable and has computable L^p-norm. Now we construct a computable sequence of dyadic step functions s_t which converges to f in L^p-norm; the key point is to show that this convergence is effective. For $t = 0, 1, 2, \ldots$ let

$$s_t(x) = 2^t \int_{w/2^t}^{(w+1)/2^t} f(x)\,dx \quad \text{for } \frac{w}{2^t} < x \le \frac{(w+1)}{2^t},\ 0 \le w < 2^t.$$

Thus, on each dyadic interval $(w/2^t, (w+1)/2^t]$, $s_t(x)$ takes the average value of $f(x)$ over this interval.

LEMMA 3. *With $\|\ \|$ denoting the L^p-norm, we have*

$$\|s_0\| \le \|s_1\| \le \|s_2\| \le \cdots \le \|f\|; \qquad 2\|s_t\| \le \|s_t + f\| \quad \textit{for all } t.$$

PROOF. We consider the L^p-norm on the subintervals $(a,b]$ of the dyadic partition. By the Hölder inequality, for any interval $(a,b]$ and any L^p-function h,

$$\left|\frac{1}{b-a}\int_a^b h(x)\,dx\right| \le \left(\frac{1}{b-a}\int_a^b |h(x)|^p\,dx\right)^{1/p}.$$

Now a simple calculation shows that if $s(x)$ is the constant function which results from averaging $h(x)$ over $(a,b]$, (i.e. $s(x) = (b-a)^{-1}\int_a^b h(x)\,dx$, $a \le x \le b$) then $\|s\|_{ab} = |s(x)|(b-a)^{1/p}$—where $\|\ \|_{ab}$ denotes the L^p-norm over $(a,b]$. Thus the above inequality yields $\|s\|_{ab} \le \|h\|_{ab}$.

Now, by definition, on each dyadic interval $(w/2^t, (w+1)/2^t]$, $s_t(x) =$ constant $=$ the average value of $f(x)$ over this interval. Then, over the same interval (where s_t is constant but s_{t+1} is not), $s_t(x) =$ the average value of $s_{t+1}(x)$. Similarly, on the same interval, $2s_t(x) =$ the average value of $s_t(x) + f(x)$.

Now the lemma follows if we consider separately each dyadic interval $(a,b] = (w/2^t, (w+1)/2^t]$ where $s_t(x)$ is constant. Using the Hölder corollary above, we let $h = f$ (to get $\|s_t\| \le \|f\|$), $h = s_{t+1}$ (to get $\|s_t\| \le \|s_{t+1}\|$), and $h = s_t + f$ (to get $2\|s_t\| \le \|s_t + f\|$).

LEMMA 4. *The p-norms $\|s_t\|$ on $[0,1]$ converge effectively to $\|f\|$ as $t \to \infty$.*

PROOF. Classical measure theory tells us that s_t converges to f in L^p-norm, whence $\|s_t\| \to \|f\|$. Now we use the assumption that $\|f\|$ is a computable real. Since the sequence of values $\|s_t\|$ is computable and converges monotonically to $\|f\|$, the convergence is effective.

LEMMA 5. *The p-norms* $\|f - s_t\| \to 0$ *effectively as* $t \to \infty$.

PROOF. Here is where we use the fact that $p > 1$, and the Clarkson inequalities (1) above. We do the case $p \geq 2$; the other case is left to the reader. From (1) we have

$$\|f - s_t\|^p \leq 2^{p-1}(\|f\|^p + \|s_t\|^p) - \|f + s_t\|^p.$$

From Lemma 3, $\|f + s_t\|^p \geq 2^p\|s_t\|^p$, so we have

$$\|f - s_t\|^p \leq 2^{p-1}(\|f\|^p - \|s_t\|^p).$$

From Lemma 4, $\|s_t\|$ converges effectively to $\|f\|$, so by the preceding inequality, $\|f - s_t\|$ converges effectively to zero.

CONCLUSION OF THE PROOF. The sequence of step functions $\{s_t\}$ is computable by Lemma 2, and by Lemma 5 converges effectively in L^p-norm to f. Hence f is (strongly) computable.

PROOF OF THEOREM 2. We give the requisite counterexample. Let a: $\mathbf{N} \to \mathbf{N}$ be a recursive function which enumerates a recursively enumerable nonrecursive set A in a one-to-one manner. Assume $0 \notin A$. For each $t > 0$, let $u_t(x)$ be the dyadic step function whose values alternate between $+1$ and -1 on the subintervals of length 2^{-t}, i.e.

$$u_t(x) = (-1)^w \quad \text{for } w/2^t < x \leq (w+1)/2^t, \ 0 \leq w < 2^t.$$

Let

$$f(x) = 1 + \sum_{t=1}^{\infty} 10^{-a(t)} u_t(x).$$

Then the L^1-norm $\|f\|_1 = 1$ is computable. By Lemma 2 , f is weakly computable in L^1, since the action of f on the dyadic intervals $(w/2^t, (w+1)/2^t]$ is computable uniformly in w and t. (Note that, for $t' > t$, the function $u_{t'}$ has zero integral over the dyadic intervals of length 2^{-t}.)

To show that f is not strongly computable in L^1 we use

LEMMA 6. *Let* σ *be any dyadic step function whose intervals are of length* $\geq 1/2^t$. *Then for all* $t^* > t$,

$$\|f - \sigma\|_1 \geq (8/9)10^{-a(t^*)}$$

PROOF. Let

$$s_t = 1 + \sum_{k=1}^{t} 10^{-a(k)} u_k(x).$$

Then the L^1-norm of the difference, $\|s_{t+1} - s_t\| = 10^{-a(t+1)}$. Actually, much more is true. For any dyadic step function σ whose intervals are of length $\geq 1/2^t$, $\|s_{t+1} - \sigma\| \geq 10^{-a(t+1)}$.

Now let t_0 be the particular value $t_0 > t$ for which $a(t')$ is minimal over all $t' > t$. Then $\|s_{t_0} - \sigma\| \geq 10^{-a(t_0)}$. By the triangle inequality

$$\|f - s_{t_0}\| \leq \sum_{\substack{t' > t \\ t' \neq t_0}} 10^{-a(t')} \leq \frac{1}{9} 10^{-a(t_0)}.$$

Hence

$$\|f-\sigma\| \geq \|s_{t_0}-\sigma\| - \|f-s_{t_0}\| \geq 10^{-a(t_0)} - (1/9)10^{-a(t_0)}.$$

This proves the lemma.

Now suppose f is (strongly) L^1-computable. Then by Lemma 1, there is a computable sequence $\{\sigma_n\}$ of dyadic step functions which converges effectively to f in L^1-norn. Define $t(n)$ as the least t such that the intervals in the step function σ_n have length $1/2^t$. Since $\{\sigma_n\}$ is computable, $t(n)$ is recursive. From Lemma 6,

$$\|f-\sigma_n\| \geq \frac{8}{9}10^{-a(t^*)} \quad \text{for all } t^* > t(n).$$

Since $\|f-\sigma_n\|$ converges effectively to zero, there is a recursive function e such that

$$\|f-\sigma_n\| \leq \frac{8}{9}10^{-N} \quad \text{for } n \geq e(N).$$

Then

$$10^{-a(t^*)} \leq 10^{-N} \quad \text{for all } t^* > t(n),\ n \geq e(N),$$
$$a(t^*) \geq N \quad \text{for all } t^* > t(e(N)).$$

Hence the set A is recursive, a contradiction.

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