PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 2, October 1984

A NOTE ON HYPONORMAL WEIGHTED SHIFTS

PENG FAN

Dedicated to the memory of my sister Jing (1958–1983)

ABSTRACT. If T is a hyponormal weighted shift (unilateral or bilateral) and if p is a polynomial, $p(T)$ may not be hyponormal.

If T is a hyponormal shift (unilateral or bilateral) and if p is a polynomial, $p(T)$ may not be hyponormal. This answers negatively Question 33 of Shields [**1**], initially raised by Hong Wha Kim. The question is: If T is a hyponormal unilateral shift and if p is a polynomial, must $p(T)$ be hyponormal?

In what follows, we shall first construct a hyponormal bilateral shift T and a polynomial p so that $p(T)$ is not hyponormal. Then, by compressing T to a proper subspace, we shall show that the resulting hyponormal unilateral shift S also possesses the property that $p(S)$ is not hyponormal.

Let T be a bilateral shift defined by

$$Te_n = \begin{cases} e_{n+1} & \text{for } n \leqslant 2, \\ 2e_{n+1} & \text{for } n \geqslant 3, \end{cases}$$

and let $p(z) = z + az^2$ for $0 < a < \sqrt{5}/5$. Since the weight sequence of T is nondecreasing, T is hyponormal. Next, an elementary computation shows that

$$[p(T)^*, p(T)] \qquad (= p(T)^*p(T) - p(T)p(T)^*) = 0_1 \oplus A \oplus 0_2,$$

where 0_1 and 0_2 are zero operators defined on $H_1 = V_{n \leqslant 1}\{e_n\}$ (the subspace spanned by $\{e_n\}_{n \leqslant 1}$) and $H_2 = V_{n=5}^{\infty}\{e_n\}$, respectively, and

$$A = \begin{bmatrix} 3a^2 & 3a & 0 \\ 3a & 3 + 15a^2 & 6a \\ 0 & 6a & 12a^2 \end{bmatrix}$$

is the matrix representation of the compression of $[p(T)^*, p(T)]$ to $V\{e_2, e_3, e_4\}$ with respect to basis $\{e_2, e_3, e_4\}$. Since $\operatorname{Det} A = 108a^4(5a^2 - 1) < 0$, $[p(T)^*, p(T)]$ is not positive. Therefore, $p(T)$ is not hyponormal as desired.

Now, let S be the compression of T to $H_0 = V_{n=0}^{\infty}\{e_n\}$. That is, $S = P_{H_0}T|_{H_0}$ in which P_{H_0} denotes the projection operator onto H_0. Clearly, S is a hyponormal

Received by the editors April 28, 1983.
1980 *Mathematics Subject Classification*. Primary 47B20, 47B37.

0002-9939/84 $1.00 + $.25 per page

unilateral shift. An almost identical computation provides us with

$$[p(S)^*, P(S)] = B \oplus A \oplus 0_2$$

where A and 0_2 are defined as above and

$$B = \begin{bmatrix} 1 + a^2 & a \\ a & a^2 \end{bmatrix}$$

is the matrix representation of the compression of $[p(S)^*, P(S)]$ to $V\{e_0, e_1\}$ with respect to basis $\{e_0, e_1\}$. An identical argument concludes the proof that $p(S)$ is not hyponormal.

References

1. A. L. Shields, *Weighted shift operators and analytic function theory*, Math. Surveys, no. 13, Amer. Math. Soc., Providence, R. I., 1974.

Department of Mathematics, Marshall University, Huntington, West Virginia 25701