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# MAXIMAL $p$-LOCALS

GEOFFREY R. ROBINSON[1]

ABSTRACT. We prove a theorem on maximal $p$-local subgroups of finite groups which has the Baer-Suzuki theorem as a corollary.

In [**1**], H. Wielandt proved that if $X$ is subgroup of the finite group $G$ and $X$ is subnormal in every maximal subgroup of $G$ containing it, then either $X$ is subnormal in $G$, or else $X$ is contained in a unique maximal subgroup of $G$. He then gave another proof of the Baer-Suzuki theorem, using this result. In this paper, we prove a theorem about maximal $p$-local subgroups of finite groups which is similar in nature to Wielandt's result, and which also leads to a short proof of the Baer-Suzuki theorem. The result presented here can probably be obtained by a modification of Wielandt's argument, but we feel that the method of proof given here is of some interest in itself.

THEOREM 1. *Let $p$ be a prime and $X$ be a $p$-subgroup of the finite group $G$. Suppose that $X \leq O_p(M)$ whenever $M$ is a maximal $p$-local subgroup of $G$ containing $X$, but that $X \not\leq O_p(G)$. Then $X$ is contained in precisely one maximal $p$-local subgroup of $G$.*

PROOF. Let $P$ be a Sylow $p$-subgroup of $G$ containing $X$. We first claim that $P$ is contained in precisely one maximal $p$-local subgroup of $G$. Suppose that $P \leq A \cap B$ where $A$ and $B$ are maximal $p$-local subgroups of $G$. Let $Y$ be the weak closure of $X$ in $P$ with respect to $G$. Then $Y \lhd A$, for any conjugate of $X$ contained in $A$ is contained in $O_p(A)$ by hypothesis, and $O_p(A) \leq P$. Likewise, $Y \lhd B$, so that $A = N_G(Y) = B$ (for $Y \ntriangleleft G$ as $X \not\leq O_p(G)$).

We next claim that each maximal $p$-local subgroup of $G$ which contains $X$ contains a Sylow $p$-subgroup of $G$. Let $A$ be a maximal $p$-local subgroup of $G$ containing $X$, and let $Q \in \mathrm{Syl}\, p(A)$. Then $X \leq Q$, as $X \leq O_p(A)$. Let $Z$ be the weak closure of $X$ in $Q$ with respect to $G$. Then $Z \lhd A$, and $Z \ntriangleleft G$, so $A = N_G(Z)$. Hence $N_G(Q) \leq A$ (for $Z \lhd N_G(Q)$) so that $Q \in \mathrm{Syl}\, p(G)$.

Suppose that the theorem is false. Then we may choose maximal $p$-local subgroups $A$ and $B$ with $X \leq A \cap B$, $A \neq B$, and $|A \cap B|_p$ as large as possible. Let $R \in \mathrm{Syl}\, p(A \cap B)$. Then $X \leq R$, as $X \leq O_p(A \cap B)$. Now $R \notin \mathrm{Syl}\, p(A)$ and $R \notin \mathrm{Syl}\, p(B)$ (for if $R \in \mathrm{Syl}\, p(A)$, then $R \in \mathrm{Syl}\, p(G)$, so $A$ is the unique maximal $p$-local subgroup of $G$ containing $R$, whereas $R \leq B$, and $A \neq B$). Let $C$ be a maximal $p$-local subgroup of $G$ containing $N_G(R)$ ($N_G(R) < G$ as $X \leq O_p(G)$ and $X \leq R$). Then $A \cap C \geq N_A(R)$; so $|A \cap C|_p > |R|$. By the maximality of $|A \cap B|_p$, we have $A = C$. Similarly, $B = C$, so $A = B$, a contradiction. The theorem is proved.

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COROLLARY 2 (BAER-SUZUKI THEOREM). *Let $X$ be a $p$-subgroup of the finite group $G$, $p$ a prime. Suppose that $\langle X, X^g \rangle$ is a $p$-group for each $g \in G$. Then $X \leq O_p(G)$.*

PROOF. We may suppose that $X \leq O_p(M)$, whenever $M$ is a maximal $p$-local subgroup of $G$ containing $X$. Suppose that $X \leq O_p(G)$. Then there is a unique maximal $p$-local subgroup of $G$ containing $X$, say $L$. For each $g \in G$, $\langle X, X^g \rangle \leq L$, as $\langle X, X^g \rangle$ is a $p$-subgroup. Hence $X^g \leq L$, so $X \leq L^{g^{-1}}$, and hence $L^{g^{-1}} = L$. Thus $g \in N_G(L).\leq N_G(O_p(L)) = L$, a contradiction, as $L < G$ and $g$ is arbitrary.

## REFERENCES

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, 5734 UNIVERSITY AVENUE, CHICAGO, ILLINOIS 60637