PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 1, January 1985

TOPOLOGICAL EQUIVALENCE IN THE SPACE OF INTEGRABLE VECTOR-VALUED FUNCTIONS

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ABSTRACT. The Banach space $L^1(0, T; X)$ is retopologized by $|||f||| = \max\|\int_a^b f\,dt\|$, $0 \leqslant a \leqslant b \leqslant T$, where $\|\cdot\|$ is the norm in the given Banach space X. It is shown here that this topology coincides with the usual weak topology of $L^1(0, T; X)$ on a wide class of weakly compact subsets.

Let X be a Banach space and $T > 0$. Denote by $L^1(0, T; X)$ the Banach space of all (Bochner) integrable functions (equivalence classes) on $[0, T]$ with the norm

$$|f|_1 = \int_0^T \|f(\tau)\|d\tau,$$

where $\|\cdot\|$ is the norm in X. $L^1(0, T; X)$ can be retopologized with the weaker norm

$$|||f||| = \max_{0 \leqslant a \leqslant b \leqslant T} \left\| \int_a^b f(\tau)\,d\tau \right\|.$$

Denote this space by $L^1(|||\cdot|||)$. Note that

$$|||f|||_1 = \max_{0 \leqslant a \leqslant T} \left\| \int_0^a f(\tau)\,d\tau \right\|$$

is an equivalent norm in $L^1(|||\cdot|||)$. The space $L^1(|||\cdot|||)$ was recently used to obtain existence results for some kinds of abstract differential equations (see e.g. [**4**], [**5**]). It was observed that subsets of the form

$$\{ f \in L^1(0, T; X) : f(t) \in K \text{ almost everywhere on } [0, T] \}$$

are compact in $L^1(|||\cdot|||)$ if the set K is compact in X. The main purpose of this note is to show that for a wide class of subsets of $L^1(0, T; X)$ (particularly of the above type), the topology generated by $|||\cdot|||$ coincides with the usual weak topology of $L^1(0, T; X)$. On the connections between $L^1(0, T; H)$ and $L^1(|||\cdot|||)$, where H is a Hilbert space, see [**5**].

DEFINITION. We say that a set $F \subset L^1(0, T; X)$ has property (U) if:

(i) F is bounded and uniformly integrable.

(ii) For every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset X$ such that for every $f \in F$ there exists a measurable set $\Omega_{f,\varepsilon}$ with $\mu([0, T] \setminus \Omega_{f,\varepsilon}) \leqslant \varepsilon$ and $f(t) \in K_\varepsilon$ for $t \in \Omega_{f,\varepsilon}$.

Received by the editors November 1, 1983, and, in revised form, February 23, 1984.

1980 *Mathematics Subject Classification*. Primary 46E40, 28B05.

Key words and phrases. Vector-valued functions, weak compactness, Banach spaces, topological equivalence.

Here μ is the Lebesgue measure on $[0, T]$. On the property (U), see [**1** and **3**].

Recall that if we denote by l_1 the Banach space of all real summable sequences $\{\xi_i\}_1^\infty$ with norm $|x|_1 = \sum_1^\infty |\xi_i|$, where $x = \{\xi_i\}_1^\infty \in l_1$, then the conjugate space m is the space of all real bounded sequences $\{\zeta_i\}_1^\infty$ with norm $|y|_\infty = \sup_i |\zeta_i|$, where $y = \{\zeta_i\}_1^\infty \in m$. Let K be a compact set. Denote by $C(K)$ the Banach space of all continuous functions on K with sup-norm $|\cdot|_\infty$. The conjugate $(C(K))^* = M(K)$ is the Banach space of all Radon measures on K. The conjugate $(L^1(0, T; X))^*$ is the space $\Lambda(0, T; X^*)$ of all essentially bounded scalar measurable functions $g: [0, T] \to X^*$, and every linear continuous functional on $L^1(0, T; X)$ is given by

$$f \to \int_0^T (f(\tau), g(\tau))\, d\tau,$$

where $f \in L^1(0, T; X)$, $g \in \Lambda(0, T; X^*)$ and $(\cdot, \cdot)$ is the pairing between X and X^*. (See [**2**, 8.14–8.18].) Our main result is the following:

THEOREM. *Let X be a Banach space and $T > 0$. Let the set $F \subset L^1(0, T; X)$ have property* (U). *Then the weak topology of $L^1(0, T; X)$ and the topology of $L^1(|||\cdot|||)$ coincide on F. Moreover, F is relatively compact in $L^1(|||\cdot|||)$.*

PROOF. Note that the set $\bigcup_{n=1}^\infty K_{1/n}$ is separable in X ($K_{1/n}$ is defined as in the Definition). Therefore F is separable in $L^1(0, T; X)$ and we can suppose without loss of generality, that X is separable.

Let $\{t_n\}_1^\infty$ be a dense sequence in $[0, T]$ and $\{y_n^*\}_1^\infty$ a weak-star dense sequence in the unit ball S^* of X^*. Let $\chi[t_n, t_m]$ be a characteristic function of the interval $[t_n, t_m]$. Then the set of all the functions of the form $y_n^* \cdot \chi[t_p, t_q]$, $t_p < t_q$, is countable. Denote these functions by $\{\phi_n\}_1^\infty$. Then

$$|||f||| = \sup_n \left| \int_0^T (f(\tau), \phi_n(\tau))\, d\tau \right|.$$

Define a linear continuous operator $P: L^1(0, T; X) \to m$ by

$$Pf = \left\{ \int_0^T (f(\tau), \phi_n(\tau))\, d\tau \right\}_{n=1}^\infty.$$

Note that $|||f||| = |Pf|_\infty$.

We will prove that $P(F) \subset m$ is relatively (norm) compact in m. Suppose there exists a compact set $K \subset X$ such that $F \subset F(K) = \{f \in L^1(0, T; X): f(\tau) \in K$ almost everywhere on $[0, T]\}$. Recall that $C(K)$ is the Banach space of continuous functions on K, and define the operator $\hat{P}: l_1 \to L^1(0, T; C(K))$ by $\hat{P}e_n = \phi_n$ on the standard basis $\{e_n\}_1^\infty$ of l_1, and then extend it by linearity and continuity to all l_1. The set $\{\phi_n\}_1^\infty \subset L^1(0, T; C(K))$ is norm compact. This can be checked directly or by using the criterion of compactness in the spaces $L^p(0, T; X)$ (see [**3**, Theorem A.1]). Thus $\hat{P}$ is compact. Its dual, $\hat{P}^*$: $(L^1(0, T; C(K)))^* \to m$, is also compact. Thus $\hat{P}^*$: $\Lambda(0, T; M(K)) \to m$,

$$P^*g = \left\{ \int_0^T (\phi_n(\tau), g(\tau))\, d\tau \right\}_{n=1}^\infty,$$

where the pairing $(\cdot,\cdot)$ is

$$(\phi_n(\tau), g(\tau)) = \int_K \phi_n(x,\tau)\, dg(x,\tau)$$

for the measure $dg(x,\tau)$ corresponding to $g(\tau)$. In particular, if $g(\tau)$ is a Dirac measure for each $\tau \in [0,T]$ and $g(\tau)$ is concentrated at $f(\tau) \in K \subset X$, then $g \in \Lambda(0,T;\ M(K))$ and

$$\hat{P}^* g = \left\{ \int_0^T (f(\tau), \phi_n(\tau))\, d\tau \right\}_{n=1}^{\infty} = Pf.$$

Thus the action of the operator $\hat{P}$ on F can be identified with the action of the operator $\hat{P}^*$, and the image $P(F)$ is relatively compact in m. Now we can suppose that F is a general set with property (U).

Consider the sets $F_\varepsilon = \{f \cdot \chi(\Omega_{f,\varepsilon}) : f \in F\}$. Here $\Omega_{f,\varepsilon}$ is a measurable set as in the Definition and $\chi(\Omega_{f,\varepsilon})$ is its characteristic function. The set F is uniformly integrable, hence for each $\delta > 0$ there exists an $\varepsilon > 0$ such that $\inf\{|f - g|_1 : g \in F_\varepsilon\} \leqslant \delta$ for every $f \in F$. Note that $|||h||| \leqslant |h|_1$ for each $h \in L^1(0,T;\ X)$. Therefore, by definition of P, we have $\inf\{|Pf - y|_\infty : y \in PF_\varepsilon\} \leqslant \delta$. But any set PF_ε is relatively compact in m, hence the set $PF \subset m$ is relatively compact. Since P: $L^1(0,T;\ X) \to m$ is continuous in the norm topologies, it is also continuous in the weak topologies. The weak and strong topologies coincide on PF. Thus the restriction $P|_F$ is continuous if we take the weak topology in $L^1(0,T;\ X)$ and the strong one in m. By [**1**, Proposition 13] any set F with property (U) is relatively weakly compact in $L^1(0,T;\ X)$. Since we can suppose that F is convex, the linear map P: $F \to PF$ is a homeomorphism in these topologies. The strong topology of m on PF is the strong topology of $L^1(|||\cdot|||)$ on F. Thus F is relatively compact in $L^1(|||\cdot|||)$ and the theorem is proved.

ACKNOWLEDGEMENT. The author is very grateful to the referee for his valuable suggestions.

REFERENCES

1. J. Bourgain, *An averaging result for l_1-sequences and application to weakly conditionally compact sets in L^1_X*, Israel J. Math. **32** (1979), 289–298.

2. R. E. Edwards, *Functional analysis*, Holt, Rinehart and Winston, New York, 1965.

3. S. Gutman, *Compact perturbations of m-accretive operators in general Banach spaces*, SIAM J. Math. Anal. **13** (1982), 789–800.

4. E. Schechter, *Evolution generated by continuous dissipative plus compact operators*, Bull. London. Math. Soc. **13** (1981), 303–308.

5. ______, *Perturbations of regularizing maximal monotone operators*, Israel J. Math. **43** (1982), 49–61.

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