PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 1, January 1985

KÖTHE'S EXAMPLE OF AN INCOMPLETE LB-SPACE

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ABSTRACT. The inductive limit of reflexive Banach spaces is fast complete. But the inductive limit of reflexive Fréchet spaces may not be fast complete. Also, an example of a complete, reflexive, nuclear space with a quotient space which is not fast complete is given.

In [**1**, §31.6] Köthe presents a sequence of Banach spaces whose inductive limit is not quasi-complete. It is shown in [**4** and **5**] that inductive limit of a sequence of reflexive Banach spaces is complete. Here we construct a sequence of quite natural nuclear Fréchet (hence reflexive) spaces E_n whose $\operatorname{ind}\lim E_n$ is not fast complete, and therefore it is neither sequentially nor quasi-complete.

As in [**3**], for an absolutely convex set A in a locally convex space X, we denote by X_A the linear hull of A equipped with the topology generated by $\{\lambda A;\ \lambda > 0\}$. If X_A is a Banach space, we call A a Banach disk. The space X is fast complete if each set bounded in X is contained in a bounded Banach disk.

Both sequential and quasi-completeness are defined in [**1**] and the latter implies the former [**1**, §18.4]. Every sequentially complete locally convex space X is fast complete.

To prove it, take a set A bounded in X and denote by B the absolutely convex closed hull of A in X. It is sufficient to show that B is a Banach disk. Let $\{x_n\}$ be a Cauchy sequence in X_B. Since B is bounded in X, the identity map $X_B \to X$ is continuous, and $\{x_n\}$ is a Cauchy sequence in X. Let $x_0 \in X$ be its limit (in the topology of X). Further, $\{x_n\}$ is bounded in X_B and therefore it is contained in λB for some $\lambda > 0$. But λB is closed in X, hence $x_0 \in \lambda B \subset X_B$, which implies the convergence of $\{x_n\}$ to x_0 in the topology of X_B.

Let $E_1 \subset E_2 \subset \cdots$ be Fréchet spaces with continuous $\mathrm{id}\colon E_n \to E_{n+1}$ for each $n \in N$, and $E = \operatorname{ind}\lim E_n$. It is proved in [**1**, §19.5(5)] that if E is quasi-complete then each set bounded in E is contained and bounded in some E_n. This result cannot be reversed. But Qiu Jing Huei showed in [**5**] that E is fast complete iff each set bounded in E is contained and bounded in some E_n.

Let $D_n = (0,\infty)\backslash\{1,2,\ldots,n\}$, $n \in N$, and let E_n be the space $\mathfrak{C}^\infty(D_n)$ with topology generated by the seminorms

$$\|f\|_{n,m} = m \cdot \sup\{\, |f^{(i)}(x)|;\ x \in [0,m],\ |x-j| \geq 1/(m+1),\ i = 0,1,\ldots,m,\ j = 0,1,\ldots,n\},$$

$m = 1,2,\ldots$. Then each E_n is a Fréchet space, $E_1 \subset E_2 \subset \cdots$, all identity maps $E_n \to E_{n+1}$ are continuous, and $E = \operatorname{ind}\lim E_n$ is Hausdorff.

Received by the editors February 15, 1983 and, in revised form, January 30, 1984.
1980 *Mathematics Subject Classification*. Primary 46A12; Secondary 46A14, 46A45, 46E10.
Key words and phrases. LB-space, LF-space, completeness, fast completeness.

Each set D_n is open, hence each E_n is the inductive limit of a sequence of L. Schwartz spaces $\mathfrak{D}_K$, where $K \subset D_n$ is compact. The spaces $\mathfrak{D}_K$ are nuclear [**2**, Chapter 3, §8], and their countable inductive limits E_n, as well as $E = \operatorname{ind\,lim} E_n$, are nuclear too. Since nuclear Fréchet spaces are Montel, each E_n is Montel and, as such, reflexive.

Let $f_n(x) = c_n(x-n)^{n-1/2}\exp(n-x)$, $x \in (0,\infty)$, $n \in N$, where $c_n > 0$ is chosen so that $\sup\{|f_n^{(i)}(x)|;\ x \in (0,\infty),\ i = 0,1,\dots,n-1\} \leq 1/n$. Since f_n is infinitely differentiable everywhere on $(0,\infty)$ but $x = n$, we have $f_n \in E_n \backslash E_{n-1}$, $E_0 = \{0\}$, $n \in N$. Put $B = \{f_n;\ n \in N\}$. Then $B \not\subset E_n$ for every $n \in N$ and a fortiori B is not bounded in E_n. Hence to prove that E is not fast complete, it is sufficient to show that B is bounded in E.

Take 0-neighborhoods U, V in E such that $U+U \subset V$. There exists $m \in N$ such that the $\|\ \|_{1,m}$-unit ball $B_{1,m} \subset U$. Fix this m and take $n > m$. There exist $k > n$ and $\lambda > 0$ such that $\lambda B_{n,k} \subset U$. Let $g_n \in \mathfrak{C}^\infty(0,\infty)$ be such that $g_n(x) = f_n(x)$ for $x \in (0,\infty)\backslash[n-1/k, n+1/k]$. Then $g_n \in E_1 \subset E_n$, $\|f_n - g_n\|_{n,k} = 0$, and

$$\begin{aligned}\|g_n\|_{1,m} &= \|f_n\|_{1,m} \\ &\leq m \cdot \sup\{|f_n^{(i)}(x)|;\ x \in (0,\infty),\ i = 0,1,\dots,n-1\} \\ &\leq m/n < 1.\end{aligned}$$

Hence $g_n \in B_{1,m}$ and $f_n = (f_n - g_n) + g_n \in U + U \subset V$. Now, for $\alpha > 1$ such that $f_k \in \alpha V$, $k = 1,2,\dots,m$, we have $B \subset \alpha V$ and B is bounded in E.

Since all E_n are complete, reflexive, and nuclear, their direct sum $\bigoplus E_n$ is complete, reflexive, and nuclear too. The mapping T: $(x_1, x_2, \dots) \mapsto \sum x_n$: $\bigoplus E_n \to \operatorname{ind\,lim} E_n$ induces an isomorphism $(\bigoplus E_n)/\ker T \to \operatorname{ind\,lim} E_n$ [**1**, §19.5]. Thus $\bigoplus E_n$ is an example of a complete, reflexive, nuclear space with a fast incomplete quotient space.

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