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COMPLETELY BOUNDED MAPS ON C^*-ALGEBRAS[1]

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ABSTRACT. In this paper we give a simpler proof of an extension theorem for completely bounded maps defined on subspaces of C^*-algebras, a new proof of a theorem of Wittstock [**8**, Satz 4.5], and a series of propositions by extending the techniques of [**7**] to work in the case of a C-bimodule action on the C^*-algebras involved.

1. Introduction. A question of Kadison [**5**] asks whether or not every bounded homomorphism from a C^*-algebra into the algebra of operators on a Hilbert space $\mathcal{L}(H)$ is similar to a $*$-homomorphism. Hadwin [**3**] has shown that a bounded unital homomorphism from a C^*-algebra into $\mathcal{L}(H)$ is similar to a $*$-homomorphism if and only if the homomorphism belongs to the span of the completely positive maps. Wittstock [**8**] and Paulsen [**6**] proved that the span of the completely positive maps from a C^*-algebra into an injective C^*-algebra is identical with the set of completely bounded maps. Together these two results prove that a bounded unital homomorphism from a C^*-algebra into $\mathcal{L}(H)$ is similar to a $*$-homomorphism if and only if it is completely bounded. Recently, Paulsen [**7**] proved that a bounded linear operator on a Hilbert space is similar to a contraction if and only if it is completely polynomially bounded. This gives a partial answer to problem 6 of [**4**]. Therefore, the set of completely bounded maps between C^*-algebras has been shown to play an important role in the study of several problems in C^*-algebras and operator theory.

2. Completely bounded maps. In this section we obtain an extension theorem and a new proof of a theorem of Wittstock for completely bounded maps.

Let A and B be C^*-algebras. By a subspace of A we mean a (not necessarily closed) complex linear subspace. Let M_n denote the C^*-algebra of complex $n \times n$ matrices. If $\mathcal{L}$ is a subspace of A and $L\colon \mathcal{L} \to B$ is a linear map, then we set $L_n = L \otimes I_n\colon \mathcal{L} \otimes M_n \to B \otimes M_n$ by

$$L_n(a \otimes b) = L(a) \otimes b.$$

The map L is positive if $L(a) \geq 0$ whenever $a \in \mathcal{L}$ and $a \geq 0$, and is completely positive if L_n is positive for all $n = 1, 2, \ldots$. L is completely bounded if $\sup_n \|L_n\|$ is finite, and we let

$$\|L\|_{\text{cb}} = \sup_n \|L_n\|.$$

If $\|L\|_{\text{cb}} \leq 1$, then L is called a complete contraction (or completely contractive). If $\mathcal{L} \subseteq A$ is a linear subspace with $\mathcal{L} = \mathcal{L}^*$ and $L\colon \mathcal{L} \to B$ is linear, then we define

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a linear map $L^*: \mathcal{L} \to B$ by

$$L^*(a) = L(a^*)^*.$$

Let $\mathrm{Re}(L): \mathcal{L} \to B$ and $\mathrm{Im}(L): \mathcal{L} \to B$ be defined by

$$\mathrm{Re}(L) = \tfrac{1}{2}(L + L^*) \quad \text{and} \quad \mathrm{Im}(L) = \tfrac{1}{2i}(L - L^*),$$

so that

$$L = \mathrm{Re}(L) + i\,\mathrm{Im}(L)$$

with $\mathrm{Re}(L)^* = \mathrm{Re}(L)$, $\mathrm{Im}(L)^* = \mathrm{Im}(L)$ and $\mathrm{Im}(L) = \mathrm{Re}(-iL)$.

DEFINITION 2.1. Let A, B, and C be unital C^*-algebras. Let C be a subalgebra of A and B with $I_C = I_A$ and $I_C = I_B$. Let $\phi: A \to B$ be a linear map. We call ϕ a C-bihomomorphism [**8**, Satz 4.2] provided

$$\phi(c_1 a c_2) = c_1 \phi(a) c_2, \quad \text{for all } c_1, c_2 \in C.$$

DEFINITION 2.2. Let A and B be C^*-algebras, and let $\psi_i: A \to B$ be completely bounded maps for $i = 1, 2$. We define $\psi_1 \leq \psi_2$ provided $\psi_2 - \psi_1$ is completely positive.

A C^*-algebra A is called injective if it has Arveson's extension property. We know that a C^*-algebra A is injective if and only if there is a completely positive projection of $\mathcal{L}(H)$ onto A [**2**]. Hence $A \otimes M_n$ is injective, for $n = 1, 2, \ldots$, when A is injective.

LEMMA 2.3. *Let A, B and C be unital C^*-algebras. Let C be a subalgebra of A and B with $I_C = I_A$ and $I_C = I_B$, and let $\phi_i: A \to B$ be C-bihomomorphisms for $i = 1, 2, 3, 4$. If the map $\psi: A \otimes M_2 \to B \otimes M_2$, defined by*

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \phi_1(a) & \phi_2(b) \\ \phi_3(c) & \phi_4(d) \end{pmatrix}$$

is completely positive, then:

(1) *ψ is a $C \oplus C$-bihomomorphism;*
(2) *ϕ_1 and ϕ_4 are completely positive C-bihomomorphisms;*
(3) *$\phi_2 = \phi_3^*$ is a completely bounded C-bihomomorphism;*
(4) *$\pm\mathrm{Re}\,\lambda\phi \leq (\phi_1 + \phi_4)/2$ for all complex numbers λ with $|\lambda| = 1$;*
(5)

$$\|\psi\|_{\mathrm{cb}}(\phi_1)_n((a_{ij})(a_{ij})^*) \geq ((\phi_2)_n((a_{ij})))((\phi_2)_n((a_{ij})))^*,$$
$$\|\psi\|_{\mathrm{cb}}(\phi_4)_n((a_{ij})^*(a_{ij})) \geq ((\phi_2)_n((a_{ij})))^*((\phi_2)_n((a_{ij})))$$

for $(a_{ij}) \in A \otimes M_n$ and $n = 1, 2, 3, \ldots$.

PROOF. Since C is a subalgebra of A, we can make $A \otimes M_2$ a $C \oplus C$-bimodule by identifying

$$c_1 \oplus c_2 = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \quad \text{for } c_1, c_2 \in C.$$

It is easy to see that (1) holds. Observe that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \psi\left(\begin{pmatrix} a & a \\ a & a \end{pmatrix}\right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \phi_1(a) & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \psi\left(\begin{pmatrix} a & a \\ a & a \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \phi_4(a) \end{pmatrix}.$$

We know that ϕ_1 and ϕ_4 are completely positive C-bihomomorphisms. Thus (2) holds. Since

$$\psi\left(\begin{pmatrix} a & a \\ a & a \end{pmatrix}\right)^* = \left(\psi\begin{pmatrix} a & a \\ a & a \end{pmatrix}\right)^*$$

and

$$\psi_n\left(\begin{pmatrix} 0 & (b_{ij}) \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & (\phi_2)_n((b_{ij})) \\ 0 & 0 \end{pmatrix}.$$

We have that

$$\phi_2 = \phi_3^* \quad \text{and} \quad \|(\phi_2)_n((b_{ij}))\| \leq \|\psi(I_{A\otimes M_2})\|\,\|(b_{ij})\|.$$

Thus (3) holds. Note that

$$\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right)\left(\begin{pmatrix} \pm\lambda & 0 \\ 0 & 1 \end{pmatrix}\right)\psi\left(\begin{pmatrix} a & a \\ a & a \end{pmatrix}\right)\left(\begin{pmatrix} \pm\overline{\lambda} & 0 \\ 0 & 1 \end{pmatrix}\right)\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right)$$
$$= \begin{pmatrix} \phi_1(a) + \phi_4(a) \pm \lambda\phi_2(a) \pm (\lambda\phi_2)^*(a) & 0 \\ 0 & 0 \end{pmatrix}$$

for all complex numbers λ with $|\lambda| = 1$, so that $\pm\mathrm{Re}\,\lambda\phi_2 \leq (\phi_1 + \phi_4)/2$. thus (4) holds. By the Schwarz inequality for completely positive maps, we have

$$\begin{aligned}
\|\psi\|_{\mathrm{cb}}\psi_n&\left(\begin{pmatrix} 0 & (a_{ij}) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & (a_{ij}) \\ 0 & 0 \end{pmatrix}^*\right) \\
&= \|\psi\|_{\mathrm{cb}}\begin{pmatrix} (\phi_1)_n((a_{ij})(a_{ij})^*) & 0 \\ 0 & 0 \end{pmatrix} \\
&\geq \left(\psi_n\left(\begin{pmatrix} 0 & (a_{ij}) \\ 0 & 0 \end{pmatrix}^*\right)\right)^*\psi_n\left(\begin{pmatrix} 0 & (a_{ij}) \\ 0 & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} ((\phi_2)_n(a_{ij}))((\phi_2)_n(a_{ij}))^* & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Similarly, we have the other inequality. Thus the lemma is proved. The following is new proof of a theorem of Wittstock [**8**, Satz 4.5]:

THEOREM 2.4. *Let A, B, and C be unital C^*-algebras with B injective. Let C be a subalgebra of A and B with $I_C = I_A$ and $I_C = I_B$, let $\mathcal{L}$ be a complex subspace of A with $c_1\mathcal{L}c_2 \subseteq \mathcal{L}$ for all $c_1, c_2 \in C$, and let $L\colon \mathcal{L} \to B$ be a completely bounded C-bihomomorphism. Then there exists a completely bounded extension $\tilde{L}\colon A \to B$ of L which is a C-bihomomorphism with $\|\tilde{L}\|_{\mathrm{cb}} = \|L\|_{\mathrm{cb}}$.*

PROOF. Without loss of generality, we may assume that $\|L\|_{\mathrm{cb}} = 1$. Let

$$S = \left\{\begin{pmatrix} c_1 & a \\ b^* & c_2 \end{pmatrix} : c_1, c_2 \in C \text{ and } a, b \in \mathcal{L}\right\},$$

and define $\phi\colon S \to B \otimes M_2$ by

$$\phi\left(\begin{pmatrix} c_1 & a \\ b^* & c_2 \end{pmatrix}\right) = \begin{pmatrix} c_1 & L(a) \\ L(b)^* & c_2 \end{pmatrix}.$$

Let ε be a fixed positive number and

$$\begin{pmatrix} H & E \\ E^* & K \end{pmatrix}$$

a positive element in $S \otimes M_n$. Then

$$\begin{pmatrix} (H+\varepsilon)^{-1/2} & 0 \\ 0 & (K+\varepsilon)^{-1/2} \end{pmatrix}\begin{pmatrix} H+\varepsilon & E \\ E^* & K+\varepsilon \end{pmatrix}\begin{pmatrix} (H+\varepsilon)^{-1/2} & 0 \\ 0 & (K+\varepsilon)^{-1/2} \end{pmatrix} \geq 0.$$

By [**2**, p. 162], we have

$$\|L_n((H+\varepsilon)^{-1/2}E(K+\varepsilon)^{-1/2})\| \leq \|(H+\varepsilon)^{-1/2}E(K+\varepsilon)^{-1/2}\| \leq 1.$$

Thus

$$\begin{aligned} &\begin{pmatrix} 1 & L_n((H+\varepsilon)^{-1/2}E(K+\varepsilon)^{-1/2}) \\ (L_n((H+\varepsilon)^{-1/2}E(K+_\varepsilon)^{-1/2}))^* & 1 \end{pmatrix} \\ &\quad = \begin{pmatrix} 1 & (H+\varepsilon)^{-1/2}L_n(E)(K+\varepsilon)^{-1/2} \\ (K+\varepsilon)^{-1/2}L_n(E)^*(H+\varepsilon)^{-1/2} & 1 \end{pmatrix} \\ &\quad \geq 0. \end{aligned}$$

Hence

$$\begin{pmatrix} H+\varepsilon & L_n(E) \\ L_n(E)^* & K+\varepsilon \end{pmatrix} \geq 0.$$

Since ε can be arbitrary small, we have

$$\begin{pmatrix} H & L_n(E) \\ L_n(E)^* & K \end{pmatrix} \geq 0.$$

Thus the map ϕ is completely positive. Since $B \otimes M_2$ is injective, we know that ϕ has an extension $\tilde{\phi}\colon A \otimes M_2 \to B \otimes M_2$ which is a unital completely positive map. Let E' be a Hermitian element in $(A \otimes M_2) \otimes M_n$ and H' a Hermitian element in $(C \oplus C) \otimes M_n$ with $H' \pm E' \geq 0$. Since $A \otimes M_2$ is a $C \oplus C$-bimodule, this implies that

$$\tilde{\phi}_n(H' \pm E') = H' \pm \tilde{\phi}_n(E') \geq 0.$$

By [**8**, Satz 4.2], $\tilde{\phi}$ is a $C \oplus C$-bihomomorphism. We now define $\tilde{L}\colon A \to B$ by setting $\tilde{L}(a)$ to be the $(1,2)$-entry of

$$\tilde{\phi}\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right).$$

For $a \in \mathcal{L}$ we have

$$\tilde{\phi}\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & L(a) \\ 0 & 0 \end{pmatrix}.$$

Hence $\tilde{L}$ does extend L. Furthermore, since $\tilde{\phi}$ is a unital completely positive map and

$$\|\tilde{L}_n((a_{ij}))\| \le \left\| \begin{pmatrix} * & \tilde{L}_n((a_{ij})) \\ * & * \end{pmatrix} \right\| = \left\| \tilde{\phi}_n \left(\begin{pmatrix} 0 & (a_{ij}) \\ 0 & 0 \end{pmatrix} \right) \right\|$$
$$\le \|\tilde{\phi}\| \, \|(a_{ij})\|,$$

we obtain

$$\|\tilde{L}_n\| \le 1.$$

Hence

$$\|\tilde{L}\|_{\mathrm{cb}} = \|L\|_{\mathrm{cb}}.$$

Finally, it is easy to see that $\tilde{L}$ is a C-bihomomorphism. Thus the theorem is proved.

THEOREM 2.5. *Let A, B, and C be unital C^*-algebras with B injective. Let C be a subalgebra of A and B with $I_C = I_A$ and $I_C = I_B$, and let $L\colon A \to B$ be a nonzero completely bounded C-bihomomorphism. Then there exist unital completely positive C-bihomomorphisms $\phi_i\colon A \to B$, for $i = 1, 2$, such that the map $\psi\colon A \otimes M_2 \to B \otimes M_2$, defined by*

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \|L\|_{\mathrm{cb}}\phi_1(a) & L(b) \\ L^*(c) & \|L\|_{\mathrm{cb}}\phi_2(d) \end{pmatrix},$$

is a completely positive $C \oplus C$-bihomomorphism.

PROOF. Let

$$S = \left\{ \begin{pmatrix} c_1 & a \\ b^* & c_2 \end{pmatrix} : c_1, c_2 \in C \text{ and } a, b \in A \right\},$$

and define $\phi\colon S \to B \otimes M_2$ by

$$\phi\left(\begin{pmatrix} c_1 & a \\ b^* & c_2 \end{pmatrix} \right) = \begin{pmatrix} c_1 & \left(\frac{L}{\|L\|_{\mathrm{cb}}} \right)(a) \\ \left(\frac{L}{\|L\|_{\mathrm{cb}}} \right)(b)^* & c_2 \end{pmatrix}.$$

As in the proof of Theorem 2.4, we may extend ϕ to be a unital completely positive $C \oplus C$-bihomomorphism $\tilde{\phi}$ on all of $A \otimes M_2$. Let $b, c^* \in A$. We know that

$$\begin{pmatrix} 0 & b \\ c^* & 0 \end{pmatrix} \in S,$$

and

$$\phi\left(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & \left(\frac{L}{\|L\|_{\mathrm{cb}}} \right)(a) \\ \left(\frac{L}{\|L\|_{\mathrm{cb}}} \right)^*(c) & 0 \end{pmatrix}.$$

Let a' be a positive element in A with $I_A \ge a'$. Then

$$0 \le \tilde{\phi}\left(\begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} \right) \le \tilde{\phi}\left(\begin{pmatrix} I_c & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} I_c & 0 \\ 0 & 0 \end{pmatrix}.$$

Let

$$\tilde{\phi}\left(\begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Then

$$0 \leq \begin{pmatrix} p & q \\ r & s \end{pmatrix} \leq \begin{pmatrix} I_C & 0 \\ 0 & 0 \end{pmatrix},$$

which implies $q = r = s = 0$. Thus

$$\tilde{\phi}\left(\begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}.$$

Since each element a in A is a finite linear combination of positive elements in A, we have

$$\tilde{\phi}\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}.$$

Similarly,

$$\tilde{\phi}\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}.$$

Now we can define

$$\tilde{\phi}\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} \phi_1(a) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\phi}\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & \phi_2(a) \end{pmatrix}.$$

Then ϕ_1 and ϕ_2 are unital completely positive maps. Thus

$$\tilde{\phi}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \phi_1(a) & \dfrac{L}{\|L\|_{\text{cb}}}(b) \\ \left(\dfrac{L}{\|L\|_{\text{cb}}}\right)^*(c) & \phi_2(d) \end{pmatrix}.$$

Let $\psi = \|L\|_{\text{cb}}\tilde{\phi}$. It is easy to see that ϕ_1 and ϕ_2 are C-bihomomorphisms. Thus the theorem is proved.

Using Theorem 2.5, we know that $\operatorname{Re} L$ and $\operatorname{Im} L$ are dominated by the same completely positive map in the following corollary.

COROLLARY 2.6. *Let A, B and C be unital C^*-algebras with B injective. Let C be a subalgebra of A and B with $I_C = I_A$ and $I_C = I_B$, and let $L\colon A \to B$ be a completely bounded C-bihomomorphism. Then there exist unital completely positive C-bihomomorphisms $\phi_i\colon A \to B$ for $i = 1, 2$, such that*

(1) $\pm\operatorname{Re}\lambda L \leq \|L\|_{\text{cb}}(\phi_1 + \phi_2)/2$ *for all complex numbers λ with $|\lambda| = 1$, and*

(2)

$$\|L\|_{\text{cb}}^2(\phi_1)_n((a_{ij})(a_{ij})^*) \geq (L_n((a_{ij})))(L_n((a_{ij})))^*,$$
$$\|L\|_{\text{cb}}^2(\phi_2)_n((a_{ij})^*(a_{ij})) \geq (L_n((a_{ij})))^*(L_n((a_{ij}))),$$

for $(a_{ij}) \in A \otimes M_n$ and $n = 1, 2, \ldots$.

PROOF. By Theorem 2.5 and Lemma 2.3, we have (1) and (2).

REFERENCES

1. W. B. Arveson, *Subalgebra of C^*-algebras*, Acta Math. **123** (1969), 141–224.
2. M. D. Choi and E. G. Effros, *Infective and operator spaces*, J. Funct. Anal. **24** (1977), 156–209.
3. D. W. Hadwin, *Dilations and Hahn decompositions for linear maps*, Canad. J. Math. **33** (1981), 826–839.
4. P. R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76** (1970), 887–933.
5. R. V. Kadison, *On the orthogonalization of operator representations*, Amer. J. Math. **77** (1955), 600–620.
6. V. I. Paulsen, *Completely bounded maps on C^*-algebras and invariant operator ranges*, Proc. Amer. Math. Soc. **86** (1982), 91–96.
7. ——, *Every completely polynomially bounded operator is similar to a contraction*, J. Funct. Anal. **55** (1984), 1–17.
8. G. Wittstock, *Ein operatorwertiger Hahn-Banach Satz*, J. Funct. Anal. **40** (1981), 127–150.
9. ——, *Extension of completely bounded C^*-module homomorphisms*, preprint.

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