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LOCAL ERGODICITY OF LINEAR CONTRACTIONS ON $C(X)$

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ABSTRACT. The linear contraction T on $C(X)$ is locally strongly ergodic if and only if it is continuously scattered and the following condition holds: the support of every extremal (with norm one), T^*-invariant measure is contained in the set $\{x: h(x) \neq 0\}$ for some continuous function h which is T-invariant on the center.

Let X be a compact Hausdorff space and T a linear contraction acting on $C(X)$, the space of real (or complex) valued continuous functions on X. Let T^* denote the adjoint of T. We write $F(T) = \{f \in C(X): Tf = f\}$ and $F_1(T^*) = \{\mu \in C(X)^*: \|\mu\| = 1,\ T^*\mu = \mu\}$. Let $M = \text{closure}\bigcup\{\text{supp}\,\mu: \mu \in F_1(T^*)\}$. Since $f|_M = 0$ implies $Tf|_M = 0$ (see [**2**]), T induces a linear contraction T_0 acting on $C(M)$. We recall that the contraction T is called strongly ergodic if there exists a projection P such that $A_n f$ converge to Pf uniformly on X ($f \in C(X)$), where $A_n = n^{-1}(I + T + \cdots + T^{n-1})$. T is called locally strongly ergodic (l.s.e.) if the operator T_0 is strongly ergodic on $C(M)$. We recall (see [**4**]) that the linear contraction T is strongly ergodic if and only if T-invariant elements separate T^*-invariant elements (i.e. if p and q are two distinct T^*-invariant elements then there exists some T-invariant element f such that $q(f) \neq p(f)$). In the Markov case ($T \geq 0$, $T1 = 1$) the necessary and sufficient conditions for local strong ergodicity were given by Sine in [**5**]. Under some additional assumptions about $F(T)$, R. E. Atalla has given in [**1**] necessary and sufficient conditions for l.s.e. of nonpositive contractions on $C(X)$. He has shown that if the contraction T on $C(X)$ is continuously scattered (i.e. the family of continuous functions, each constant on the supports of extremal measures in $F_1(T^*)$, is sufficient to separate supports of orthogonal measures in $\text{ex}\,F_1(T^*)$) and there exists a T-invariant function h, such that $h \neq 0$ on M, then T is l.s.e. In this note we obtain a stronger version of Atalla's result. In order to get local strong ergodicitv of a linear contraction on $C(X)$ we need only the scattering assumption and the following condition: for every extremal measure m in $F_1(T^*)$ there exists a function h_m in $F(T_0)$ such that $h_m \neq 0$ on $\text{supp}\,m$. Moreover we show that for l.s.e. our condition is also necessary. Now we recall two lemmas from [**1** and **2**].

LEMMA A1 (SEE [**1**]). *Let $\varphi_m = dm/d|m|$ (the Radon-Nikodym derivative). Suppose that f is in $F(T)$ and m is in $\text{ex}\,F_1(T^*)$. Then*

$$f = \left(\int f\,dm\right)\overline{\varphi}_m, \qquad |m|\text{-a.e.}$$

LEMMA A2 (SEE [**2**]). *If m is in $\text{ex}\,F_1(T^*)$, then for every x from $\text{supp}\,m$,*

$$\text{supp}\,T^*\delta_x \subset \text{supp}\,m.$$

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By Z_0 we denote the set $\bigcup \operatorname{supp} m$ where the union is taken over all extremal m in $F_1(T^*)$. The set $\{x \in M\colon (\exists f)_{F(T_0)}\, f(x) \neq 0\}$ will be denoted by Z.

THEOREM. *Let T be a linear contraction on $C(X)$. Then T is l.s.e. if and only if the following conditions hold:*
(i) $Z_0 \subset Z$;
(ii) T *is continuously scattered.*

PROOF. We can assume, without loss of generality, that $M = X$ (so we have $T = T_0$). Let T be strongly ergodic and $m \in \operatorname{ex} F_1(T^*)$. By the Sine separating theorem we can find a T-invariant continuous function f such that $\int f\,dm \neq 0$ (see [**4**]). From Lemma A1 $f = (\int f\,dm)\overline{\varphi}_m$, $|m|$-a.e. and thus $|f| = |\int f\,dm| \neq 0$ on $\operatorname{supp} m$, so we obtain condition (i). Let ν and λ be two orthogonal measures from $\operatorname{ex} F_1(T^*)$. By [**4**] there exists $f \in F(T)$ such that $\int f\,d\nu \neq \int f\,d\lambda = 0$. By Lemma A1 the function $|f|$ is constant on the support of each extreme measure from $F_1(T^*)$. Also on $\operatorname{supp}\nu$ we have $|f|(x) = |\int f\,d\nu| \neq 0$, while on $\operatorname{supp}\lambda$ we have $|f|(x) = |\int f\,d\lambda| = 0$. Thus $|f|$ separates $\operatorname{supp}\nu$ from $\operatorname{supp}\lambda$.

Conversely, suppose that (i) and (ii) are satisfied. At first we observe that for every function $h \in F(T)$, the sets $Y_{h,\varepsilon} = \{x\colon |h(x)| \geq \varepsilon\}$ are T-invariant (i.e. if $g = 0$ on $Y_{h,\varepsilon}$ then $Tg = 0$ on $Y_{h,\varepsilon}$). In fact, if $h \in F(T)$ then, by Lemma A1, $|h|$ is constant on $\operatorname{supp} m$ whenever m is extreme in $F_1(T^*)$ and, by Lemma A2, $\operatorname{supp} m$ is T-invariant. Further, each such support set is either contained in $Y_{h,\varepsilon}$ or disjoint from it. It follows easily that such support sets are dense in $Y_{h,\varepsilon}$, so the latter set is T-invariant. Thus by Atalla's theorem from [**1**], for every continuous function f on X, $\lim_n A_n f(x)$ exists on Z and this limiting function (denoted by $\overline{f}$) is continuous on Z. We show now that $A_n f(x)$ converges to 0 for $x \in X\backslash Z$ and

$$\lim_{x_\alpha \to x} \overline{f}(x_\alpha) = 0 \quad (x_\alpha \in X).$$

To prove this we observe that if $x \in X\backslash Z$ then, for every $h \in F(T)$ and $\varepsilon > 0$, $|T^*\delta_x|(Y_{h,\varepsilon}) = 0$. Therefore for every $h \in F(T)$ we have $|T^*\delta_x|(\{|h| > 0\}) = 0$ and thus $\operatorname{supp}|T^*\delta_x| \subset X\backslash Z$. It is clear now that the closed set $X\backslash Z$ is T-invariant. Hence by [**3**], for every $f \in C(X)$ we have

$$\lim_n \sup_{x \in Z^c} |A_n f(x)| = \sup_m \int f\,dm$$

where the sup is taken over all T^*-invariant measures with norm one which are concentrated on $X\backslash Z$. But by assumption (i), there are no T^*-invariant measures concentrated on $X\backslash Z$. So $A_n f(x)$ converge to 0 uniformly for $x \in X\backslash Z$, and now we only have to show that $\lim_{x_\alpha \to x} \overline{f}(x_\alpha) = 0$ for every $x_\alpha \to x \in X\backslash Z$. Without loss of generality we can assume that $x_\alpha \in \operatorname{supp} m_\alpha$ where $m_\alpha \in \operatorname{ex} F_1(T^*)$. By Lemma A1 it is clear that $|\overline{f}(x_\alpha)| = |\int f\,dm_\alpha|$, so $\lim_{x_\alpha \to x} \overline{f}(x_\alpha) = 0$ since the net of measures m_α converges to 0 in the weak* topology (there are no T^*-invariant measures concentrated on $X\backslash Z$).

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