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$\aleph_0$-POINT COMPACTIFICATIONS OF LOCALLY COMPACT SPACES AND PRODUCT SPACES

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ABSTRACT. We give necessary and sufficient conditions for a locally compact space to have a compactification with countably infinite remainder. We also characterize the product space of two locally compact spaces having such a compactification.

1. Introduction. All spaces are assumed to be completely regular and T_1, and we denote the set of positive integers by N.

A compactification αX of a space X is called an $\aleph_0$-point compactification (we say X has an $\aleph_0$CF) if the remainder $\alpha X - X$ has cardinality $\aleph_0$.

We consider some characterizations of a locally compact space with an $\aleph_0$CF. The first to characterize such spaces was Magill [**7**]. In §2 we give a characterization analogous to a theorem of Magill [**6**] concerning the n-point compactification.

The notion of singular set was introduced by Cain [**1**], and many authors (see for example [**2**–**4**]) have investigated the relationships between compactifications and singular sets. In §3, as a corollary of our theorem, we characterize a locally compact space having an $\aleph_0$CF by singular sets.

Recently, Hoshina [**5**] characterized a product space of two paracompact spaces having a compactification with countable remainder. Here countable means finite or infinite countable. In §4, as an application of our theorem, we characterize a product space of two locally compact spaces having an $\aleph_0$CF.

I wish to thank Professor T. Hoshina for his helpful suggestions, especially for the simple proof of Theorem 1.

2. $\aleph_0$-point compactifications of locally compact spaces. We begin with

LEMMA 1. *Let A be a countably infinite subset of a space X. Then there are a countably infinite subset $B = \{b_i | i \in N\}$ of A and a sequence $\{U_i | i \in N\}$ of pairwise disjoint open subsets of X such that $b_i \in U_i$ and $\mathrm{Bd}_X U_i \cap A = \varnothing$ for each $i \in N$.*

PROOF. Since X is regular, there are a point $b_1 \in A$ and an open subset V_1 of X such that $b_1 \notin \mathrm{Cl}_X V_1$ and $|V_1 \cap A| = \aleph_0$. Similarly, there are a point $b_2 \in V_1 \cap A$ and an open subset V_2 of V_1 such that $b_2 \notin \mathrm{Cl}_X V_2$ and $|V_2 \cap A| = \aleph_0$. Continuing in this manner, we obtain a subset $B = \{b_i | i \in N\}$ of A and a sequence $\{V_i | i \in N\}$ of open subsets of X. Now let $W_i = V_{i-1} - \mathrm{Cl}_X V_i$ for each $i \in N$, where $V_0 = X$. By complete regularity of X, there is a continuous mapping $f_i \colon X \to I = [0, 1]$ such that $f_i(b_i) = 0$ and $f_i(X - W_i) = \{1\}$. Then there is a real number $r_i \in I$ such that

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$r_i \notin f_i(A)$. Let $U_i = f_i^{-1}([0, r_i))$. Since $\mathrm{Bd}_X U_i \subset f_i^{-1}(r_i)$, we have $\mathrm{Bd}_X U_i \cap A = \emptyset$. Hence $\{U_i | i \in N\}$ has all the required properties.

An open set U of a space X is γ-open if $\mathrm{Bd}_X U$ is compact. A space X is called rim-compact if it has a base consisting of γ-open sets. Clearly, locally compact spaces are rim-compact. The following lemma, proved by Terada [**8**, Lemma 3], is useful.

LEMMA 2. *Let X be a rim-compact space and γX the Freudenthal compactification of X. If U is γ-open in X, then*

$$\mathrm{Cl}_{\gamma X} U \cap (\gamma X - X) = (\gamma X - \mathrm{Cl}_{\gamma X}(X - U)) \cap (\gamma X - X),$$
$$(\gamma X - \mathrm{Cl}_{\gamma X}(X - U)) \cup (\gamma X - \mathrm{Cl}_{\gamma X} U) \supset \gamma X - X, \quad \textit{and}$$
$$(\gamma X - \mathrm{Cl}_{\gamma X}(X - U)) \cap (\gamma X - \mathrm{Cl}_{\gamma X} U) = \emptyset$$

hold.

We are now in a position to establish one of our main theorems.

THEOREM 1. *A locally compact space X has an* $\aleph_0$CF *if and only if there is a sequence $\{U_i | i \in N\}$ of pairwise disjoint γ-open subsets of X each with noncompact closure.*

PROOF. Let Y be an $\aleph_0$CF of X. By Lemma 1 there are a countably infinite subset $\{a_i | i \in N\}$ of $Y - X$ and a sequence $\{W_i | i \in N\}$ of pairwise disjoint open subsets of Y such that $a_i \in W_i$ and $\mathrm{Bd}_Y W_i \cap (Y - X) = \emptyset$ for each $i \in N$. Now let $U_i = W_i \cap X$ for each $i \in N$. Then $\mathrm{Cl}_X U_i$ is noncompact. Since $\mathrm{Bd}_X U_i \subset \mathrm{Bd}_Y W_i \subset X$, $\mathrm{Bd}_X U_i$ is compact. Hence $\{U_i | i \in N\}$ has all the required properties.

Conversely, let $\{U_i | i \in N\}$ be a family of pairwise disjoint γ-open subsets of X each with noncompact closure. Let γX be the Freudenthal compactification of X, $F_i = \mathrm{Cl}_{\gamma X} U_i \cap (\gamma X - X)$ and $F_0 = (\gamma X - X) - \bigcup\{F_i | i \in N\}$. Then $\mathcal{F} = \{F_i | i = 0, 1, 2, \ldots\} \cup \{\{x\} | x \in X\}$ is an upper semicontinuous decomposition of γX.

It is obvious that $\mathcal{F}$ is a decomposition of γX. Hence it suffices to show the upper semicontinuity of $\mathcal{F}$.

Case 1. If $x \in X$, then for any open neighborhood U of x in γX, $W = U \cap X$ is an open neighborhood of x in γX. And for any element F' of $\mathcal{F}$ such that $F' \cap W \neq \emptyset$, we have $F' \subset W \subset U$.

Case 2. Let F be an element of $\{F_i | i \in N\}$ and U an open subset of γX containing F. By Lemma 2, F is open in $\gamma X - X$. Thus there is an open subset V of γX such that $F = V \cap (\gamma X - X)$. Let us set $W = U \cap V$. Then for any element F' of $\mathcal{F}$ such that $F' \cap W \neq \emptyset$, we have $F' \subset W \subset U$. Clearly, $F \subset W \subset U$.

Case 3. Let U be an open subset of γX containing F_0. Let $\mathcal{F}'$ be the collection of all elements of $\{F_i | i \in N\}$ not contained in U. Since $\gamma X - X$ is compact and F_i is open in $\gamma X - X$, $\mathcal{F}'$ is finite. Thus $W = U - \bigcup\{F | F \in \mathcal{F}'\}$ is open in γX. For any element F' of $\mathcal{F}$ such that $F' \cap W \neq \emptyset$, we have $F' \subset W \subset U$. Clearly, $F_0 \subset W \subset U$.

Hence $\mathcal{F}$ is an upper semicontinuous decomposition of γX. Let $Y = \gamma X / \mathcal{F}$ be the quotient space of γX determined by the upper semicontinuous decomposition $\mathcal{F}$. Then Y is an $\aleph_0$CF of X. Theorem 1 is proved.

3. $\aleph_0$-point compactifications and singular sets. Cain [**1**], [**2**] defined the singular set as follows: If $f\colon X \to Y$ is a continuous mapping of one locally compact space onto another, then the singular set $S(f)$ is the collection of all points $p \in Y$ such that in every neighborhood of p there is a compact set with noncompact inverse image. As a corollary of his main theorem [**2**, Theorem 3.1], Cain proved that a connected locally compact space X has an n-point compactification if and only if there is a continuous mapping f of X onto a compact space Y so that $S(f)$ consists of exactly n points [**2**, Theorem 3.2]. Cain, Chandler and Faulkner [**3**] strengthened this theorem, so it remains true without connectedness. Chandler and Tzung [**4**] defined the remainder induced by f to be

$$C(f) = \bigcap \{\mathrm{Cl}_Y f(X - F) | F \text{ is a compact subset of } X\},$$

and proved that (whenever Y is compact) there is a compactification αX with $\alpha X - X$ homeomorphic to $C(f)$. Their "remainder induced by f" is the same as a singular set [**3**]. Thus the "if" part of Cain's theorem holds for any cardinality of $S(f)$. However, for every compactification αX there need not be a singular set $S(f)$ with $S(f)$ homeomorphic to $\alpha X - X$. Indeed, the Stone-Čech compactification βN of a countably infinite discrete space N has no singular set $S(f)$, with $S(f)$ homeomorphic to $\beta N - N$, because the cardinality of $S(f)$ is at most countable. As a corollary of Theorem 1, we can prove the following result, which is analogous to Cain's theorem.

THEOREM 2. *A locally compact space X has an $\aleph_0$CF if and only if there is a continuous mapping f of X onto a compact space Y such that $S(f)$ has cardinality $\aleph_0$.*

PROOF. It suffices to prove the "only if" part. Suppose X has an $\aleph_0$CF. By Theorem 1 there is a sequence $\{U_i | i \in N\}$ of pairwise disjoint γ-open subsets of X each with noncompact closure. For each $i = 2, 3, \ldots$, choose a point $x_i \in U_i$ and put $F_i = (\mathrm{Cl}_{\gamma X} U_i \cap (\gamma X - X)) \cup \{x_i\}$, where γX is the Freudenthal compactification of X. Let $F_1 = \gamma X - \bigcup \{\gamma X - \mathrm{Cl}_{\gamma X}(X - U_i) | i = 2, 3, \ldots\}$. Let $\mathcal{F} = \{F_i | i \in N\} \cup \{\{x\} | x \in \gamma X - \bigcup \{F_i | i \in N\}\}$. Then as in Theorem 1 we can prove that $\mathcal{F}$ is an upper semicontinuous decomposition of γX. Thus $Y = \gamma X / \mathcal{F}$ is a compact Hausdorff space. Let $g\colon X \to \gamma X$ be an embedding and $q\colon \gamma X \to Y$ a quotient mapping. Then the composition $f = q \circ g$ is a continuous mapping of X onto Y. Let us set $\{a_i\} = q(F_i)$ for each $i \in N$ and $A = \{a_i | i \in N\}$. Then we have $q^{-1}(A) = \gamma X - \bigcup \{U_i - \{x_i\} | i = 2, 3, \ldots\}$. Since X is locally compact, it is open in γX. Thus $q^{-1}(A)$ is closed in γX. Since q is a quotient mapping, A is closed in Y. Then $U = Y - A$ is open in Y, and for every compact subset K of U, $f^{-1}(K)$ is compact, because the restriction of $f\colon X \to Y$ to $U \subset Y$, $f_U\colon f^{-1}(U) \to U$, is a homeomorphism. Thus for every $y \in U$ we have $y \notin S(f)$. Next, for each a_i and any neighborhood V_i of a_i there is an open subset W_i such that $a_i \in W_i \subset \mathrm{Cl}_Y W_i \subset V_i$. Then $\mathrm{Cl}_Y W_i$ is compact but $f^{-1}(\mathrm{Cl}_Y W_i)$ is noncompact. Thus we have $a_i \in S(f)$. Hence $S(f)$ has cardinality $\aleph_0$. Theorem 2 is proved.

4. $\aleph_0$-point compactifications of product spaces. Throughout this section we assume all spaces to be locally compact. From Theorem 1 it follows that the class of all spaces having an $\aleph_0$CF is an inverse invariant of perfect mappings. Hence if X is compact and Y has an $\aleph_0$CF, then $X \times Y$ also has an $\aleph_0$CF. However, this class is not an invariant of perfect mappings. Indeed, let $Z = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $X = Z \times R$ and $Y = R$, where R denotes the real line. The projection $p\colon X \to Y$ is perfect and X has an $\aleph_0$CF. However, Y has no $\aleph_0$CF. As for the existence of an $\aleph_0$CF of a product space, we obtain the following theorem.

THEOREM 3. *A product space $X \times Y$ has an $\aleph_0$CF if and only if one of the following conditions is satisfied:*

(a) *either X or Y has an infinite number of components and the other is noncompact;*

(b) *either X or Y has a compact component and the other has an $\aleph_0$CF.*

The following corollary is obvious.

COROLLARY. *Let X and Y be connected. Then $X \times Y$ has an $\aleph_0$CF if and only if either X or Y is compact and the other has an $\aleph_0$CF.*

To prove Theorem 3 we need some lemmas.

LEMMA 3. *If X and Y are noncompact connected spaces, then $X \times Y$ has no $\aleph_0$CF.*

PROOF. Suppose $X \times Y$ has an $\aleph_0$CF. By Theorem 1 there is a sequence $\{U_i | i \in N\}$ of pairwise disjoint γ-open subsets of $X \times Y$ each with noncompact closure. Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the projections, and let (i, j) be a pair of distinct positive integers. We set

$$A = p_X(\mathrm{Bd}_{X\times Y} U_i) \times p_Y(\mathrm{Bd}_{X\times Y} U_i).$$

Then A is compact. Since X and Y are noncompact and connected, $Z = X \times Y - A$ is connected. Since $\mathrm{Bd}_{X\times Y} U_i \subset A$, we have $\mathrm{Bd}_{X\times Y} U_i \cap Z = \emptyset$. Thus $Z = (U_i \cap Z) \cup ((X \times Y - \mathrm{Cl}_{X\times Y} U_i) \cap Z)$. Hence, $Z \subset \mathrm{Cl}_{X\times Y} U_i$ or $U_i \cap Z = \emptyset$. Suppose $Z \subset \mathrm{Cl}_{X\times Y} U_i$. Since $\mathrm{Cl}_{X\times Y} U_i \cap U_j = \emptyset$, we have $U_j \subset A$. This implies that $\mathrm{Cl}_{X\times Y} U_j$ is compact, and we have a contradiction. Suppose $U_i \cap Z = \emptyset$. Then $U_i \subset A$. This implies that $\mathrm{Cl}_{X\times Y} U_i$ is compact, and we have a contradiction. Hence $X \times Y$ has no $\aleph_0$CF.

LEMMA 4. *If X is a compact connected space and Y has no $\aleph_0$CF, then $X \times Y$ has no $\aleph_0$CF.*

PROOF. Suppose $X \times Y$ has an $\aleph_0$CF. By Theorem 1 there is a sequence $\{U_i | i \in N\}$ of pairwise disjoint γ-open subsets of $X \times Y$ each with noncompact closure. Let $p_Y : X \times Y \to Y$ be the projection. Now we set $V_i = Y - p_Y(X \times Y - U_i)$. Since X is compact, p_Y is closed, therefore V_i is open in Y. Assume that $\mathrm{Cl}_Y V_i$ is compact. Then $A = X \times (\mathrm{Cl}_Y V_i \cup p_Y(\mathrm{Bd}_{X\times Y} U_i))$ is compact. Take a point $(x, y) \in U_i$. If $X \times \{y\} \subset U_i$, then $y \in V_i$. Thus $(x, y) \in A$. If $X \times \{y\} \not\subset U_i$, then $X \times \{y\} \cap \mathrm{Bd}_{X\times Y} U_i \neq \emptyset$, because $X \times \{y\}$ is connected. Thus $y \in p_Y(\mathrm{Bd}_{X\times Y} U_i)$, therefore $(x, y) \in A$. Hence $U_i \subset A$. However, this implies that $\mathrm{Cl}_{X\times Y} U_i$ is compact, and we have a contradiction. Thus $\mathrm{Cl}_Y V_i$ is noncompact. Clearly, for every pair of distinct positive integers i and j, $V_i \cap V_j = \emptyset$. Hence $\{V_i | i \in N\}$ satisfies the conditions of Theorem 1. Thus Y has an $\aleph_0$CF; however, this is a contradiction.

By Theorem 1 it is easy to prove the following lemma.

LEMMA 5. *Let X be a space which can be represented as the finite topological sum $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$. Then X has an $\aleph_0$CF if and only if there is a positive integer $(1 \leq)\ i\ (\leq n)$ such that X_i has an $\aleph_0$CF.*

PROOF OF THEOREM 3. If both X and Y have at most a finite number of components, then by Lemmas 3–5, condition (b) holds. If both X and Y have an infinite number of components, then (a) holds. Assume that X has an infinite number of components and Y has at most a finite number of components. Let $\{Y_1, Y_2, \ldots, Y_n\}$ be the collection of all components of Y. If Y is compact, then, by Lemma 5, there is a positive integer $(1 \leq)\ i\ (\leq n)$ such that $X \times Y_i$ has an $\aleph_0$CF. By Lemma 4, X has an $\aleph_0$CF. Hence (b) holds.

Conversely, let X and Y satisfy (a). Suppose X has an infinite number of components and Y is noncompact. Then there is a sequence $\{U_i | i \in N\}$ of pairwise disjoint open-and-closed subsets of X. Now let $V_i = U_i \times Y$ for each $i \in N$. Then $\{V_i | i \in N\}$ satisfies the conditions of Theorem 1. Hence $X \times Y$ has an $\aleph_0$CF.

Next, let X and Y satisfy condition (b). Suppose X has a compact component X_0 and Y has an $\aleph_0$CF. If X has an infinite number of components, then X and Y satisfy (a). Hence $X \times Y$ has an $\aleph_0$CF. If X has at most a finite number of components, then X_0 is open-and-closed in X. As described in the first part of §4, $X_0 \times Y$ has an $\aleph_0$CF. Since $X \times Y = (X_0 \times Y) \oplus ((X - X_0) \times Y)$, and by Lemma 5, $X \times Y$ has an $\aleph_0$CF. Theorem 3 is proved.

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