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TWO PROOFS IN COMBINATORIAL NUMBER THEORY

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ABSTRACT. The aim of this paper is to present a short combinatorial proof of a theorem of P. Erdős on multiplicative bases of integers. A solution of a problem of P. Erdős and D. J. Newman is also presented.

I. Let X be a set of positive integers. We say that X is a *multiplicative base* if for every positive integer n there are $x, y \in X$ such that $n = xy$. The following was proved by P. Erdős (see [**1**]).

THEOREM 1. *Let X be a multiplicative base. Then for every positive integer p there exists a positive integer n such that n can be expressed as the product of two elements of X in at least p different ways.*

In the proof of Theorem 1 we shall need the well-known theorem of Ramsey: *Let p be a given integer and let $[A]^p = C_1 \cup C_2$ be a partition of the set of all p-tuples of elements of an infinite set A into two parts. Then there exists i and an infinite set $B \subseteq A$ such that $[B]^p \subseteq C_i$. The set B is called homogeneous with respect to the partition (C_1, C_2).*

PROOF OF THEOREM 1. In the following we shall consider the integers which are products of distinct primes only. Such integers can be identified in a natural way with finite subsets of the set of all primes. Thus it suffices to prove the following

THEOREM 1′. *Let A be an infinite set. Denote by $[A]^{<\omega}$ the set of all finite subsets of A. Let $\mathscr{A} \subseteq [A]^{<\omega}$ be a set of finite subsets of A such that the following holds:*

$$(*)\qquad \begin{array}{l} \textit{For every } P \in [A]^{<\omega} \textit{ there are } Q, Q' \in \mathscr{A} \textit{ such that} \\ Q \cup Q' = P \textit{ and } Q \cap Q' = \varnothing. \end{array}$$

Then for every integer p there is a set P which can be expressed in at least p-different ways as a union of two disjoint elements of $\mathscr{A}$.

Proof of Theorem 1′ is a straightforward consequence of Ramsey's theorem: *Let p be a given positive integer, $p \geqslant 2$. For every $i = 1, \ldots, p-1$ consider a partition $[A]^i = C_1^i \cup C_2^i$ defined by $Q \in C_1^i$ iff $Q \in \mathscr{A}$. Let B be an infinite set which is homogeneous with respect to all partitions C_1^i, C_2^i, $i = 1, \ldots, p-1$ (such a set clearly*

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exists by iterating the Ramsey theorem). *From* (*) *we get that there is an* i, $1 \leqslant i \leqslant p-1$, *such that*

$$[B]^i \subseteq C_1^i \quad \textit{and} \quad [B]^{p-i} \subseteq C_1^{p-i},$$

and hence every $P \in [B]^p$ *can be represented as a union of at least* $\binom{p}{i} \geqslant p$ *elements of* $\mathscr{A}$.

REMARK. Note that the above proof gives nothing concerning the additive version of Theorem 1. This is an old problem of P. Erdős:

Problem. Let X be a set of positive integers with the property that for every positive integer n there are $x, y \in X$ such that $n = x + y$. Is is true that for every positive integer p there exists a positive integer n such that n can be expressed as the sum of two elements of X in at least p different ways?

II. Let X be a set of positive integers. We say that X is a B_2^k-sequence if the number of representations of n as the sum $x + y$, $x, y \in X$, is at most k and for some n is exactly k. The following is known as the Erdős-Newman problem:

Is it true that given a k there exists a $B_2^{(k)}$-sequence X such that for every partition $X = X_1 \cup \cdots \cup X_r$ into a finite number of parts one of the parts is a $B_2^{(k)}$-sequence?

For certain k the affirmative solution is given in [**2**]. We prove here (by different methods)

THEOREM 2. *For every* $k \geqslant 2$ *there exists a set of integers* X *such that*:

(1) X *is a* $B_2^{(k)}$*-sequence*;

(2) *For every partition of* X *into a finite number of parts one of the parts is a* $B_2^{(k)}$*-sequence*.

The proof is based on the existence of Ramsey graphs of a special type. Before stating this result we introduce some necessary notions. An ordered graph is a graph (V, E) with a (fixed) ordering of its vertices. K_3 is the complete graph with 3 vertices (i.e. the triangle). Denote by L_k the graph (V, E) defined as follows:

$$V = \{0, 1, \ldots, k+1\},$$
$$E = \{\{0, i\}; i = 1, \ldots, k\} \cup \{\{i, k+1\}; i = 1, \ldots, k\}.$$

The graph L_k is always considered as an ordered graph with the natural ordering of its vertices. (The graph L_k is depicted in Figure 1.) Let (V, E) and (V', E') be ordered graphs. We say that (V, E) is *contained* in (V', E') if $V \subseteq V'$ and $E \subseteq E'$ and the ordering of V' restricted to the set V coincides with that of V. We shall use the following

LEMMA. *For every* $k, r \geqslant 2$ *there exists an ordered graph* $G_r^k = (V, E)$ *with the following properties*:

(1) G_r^k *does not contain* L_{k+1} *and* K_3;

(2) *for every partition* $E = E_1 \cup \cdots \cup E_r$ *one of the classes* E_i *contains* L_k.

(A proof of this Lemma is too nontrivial (and lengthy) to be included here. However it follows easily from the "partite construction" which is introduced and studied in [**3**].)

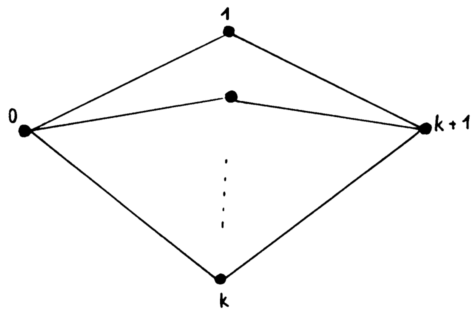

FIGURE 1

PROOF OF THEOREM 2. Put $G = \bigcup(G_r^k;\ r \geqslant 2)$—the disjoint union of graphs described in the Lemma. Consider $G = (V, E)$ as an ordered graph which contains every G_r^k, $r \geqslant 2$. Assume without loss of generality that the vertices of G are integers. Define the set X as the set of all sums $\sum 3^n$, where the summation is taken over the set I_{uv} of all positive integers n which satisfy $u \leqslant n < v$ for an edge $\{u, v\} \in E$. In the sequel the sets of the form I_{uv} are called *intervals*. We prove that X has the desired properties:

ad 1. Observe that if $z = x + y$ for $x, y \in X$ and if $z = \sum_{i \in I} \varepsilon_i 3^i$ is the triadic expansion of z (i.e. $\varepsilon_i = 1$ or 2), then I is either an interval or union of two intervals. Moreover, if $\varepsilon_i = 2$ for an $i \in I$, then I is an interval and $I_1 = \{i;\ \varepsilon_i = 2\}$ is also an interval. Therefore we may distinguish three cases which can be visualized as follows:

(i) $I_1 \neq \varnothing$. Either:

(ii) $I_1 = \varnothing$, I is an interval:

(iii) I is the union of two intervals, but it is not an interval itself:

In (i) there are at most two (depicted) possibilities for $z = x' + y'$, $x', y' \in X$. In (ii), there are at most k possibilities for the solution $z = x' + y'$ by assumption (1) of the Lemma on graphs G_r^k (graphs G_r^k do not contain L_{k+1}). In (iii), $z = x + y$ is the unique solution. Therefore X is a $B_2^{(k)}$-sequence.

As every partition of X corresponds to a partition of the edges of the graph G, the second half of the statement of Theorem 2 follows immediately from property (2) of the Lemma on the graphs G_r^k.

The methods described above enable the following generalizations of Theorems 1 and 2. Let X be a set of positive integers. Let us call X the *multiplicative j-base* if for every positive integer n there are $x_1, \ldots, x_j \in X$ such that $n = x_1 \cdots x_j$.

THEOREM I. *Let X be a multiplicative j-base. Then for every positive integer p there exists a positive integer n such that n can be expressed as the product of j elements of X in at least p different ways.*

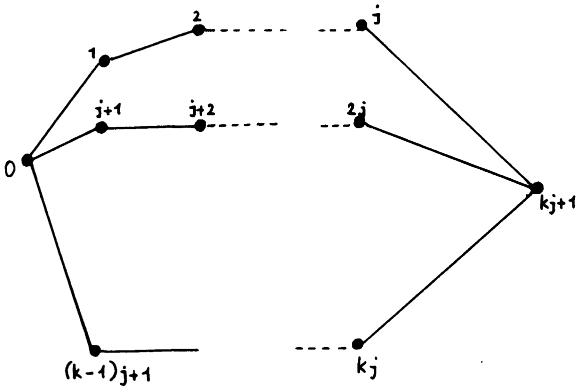

FIGURE 2

Similarly, let us say that X is a $B_j^{(k)}$-sequence if the number of representation of n as the sum $n = x_1 + x_2 + \cdots + x_i$, $i \leqslant j$, $x_1, x_2, \dots, x_i \in X$, is at most k and for some n exactly k. P. Erdős asked whether an analogy of Theorem 2 can be proved for $B_j^{(k)}$-sequences. If we use a suitable modification of graphs L_k, Theorem 2 can be strengthened as follows:

THEOREM II. *For every $k \geqslant 2$ and $j \geqslant 2$ there exists a set of integers X such that*

(1) *X is a $B_j^{(k)}$ sequence*;

(2) *for every partition of X into a finite number of parts one of the parts is a $B_j^{(k)}$-sequence.*

The proof is analogous to the above proof of Theorem 2 with the difference that instead of graphs L_k we use graphs $L_k^j = (V, E)$ defined as follows:

$$V = \{0, 1, \dots, kj + 1\},$$

$$E = \{\{0, j \cdot i + 1\}; i = 0, \dots, k - 1\}$$

$$\cup \{\{j \cdot i + l, ji + l + 1\}, i = 0, 1, \dots, k - 1, l = 1, 2, \dots, j - 1\}$$

$$\cup \{\{lj, kj + 1\}; l = 1, \dots, k\}$$

(see Figure 2).

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