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AN ARITHMETIC PROPERTY OF THE TAYLOR COEFFICIENTS OF ANALYTIC FUNCTIONS WITH AN APPLICATION TO TRANSCENDENTAL NUMBERS

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ABSTRACT. We extend a result of Popken concerning the numerators of the Taylor coefficients of algebraic functions and combine it with a result of Mahler on lacunary power series to prove an extension of a special case of a result of Cohn on the transcendence of functional values of lacunary power series evaluated at rational points.

1. Introduction. In 1959 Popken [**4**] proved the following result [**4**, Theorems 1, 2, p. 203] for numerators of Taylor coefficients of algebraic functions:

POPKEN'S THEOREM. *Let the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, with rational coefficients a_n $(n = 0, 1, 2, \ldots)$ and convergent in a neighbourhood $|z| < R$ of the origin, represent a branch of an algebraic function which is not a polynomial. Let b denote a rational number such that $0 < |b| < R$. Let $S_n = \sum_{\nu=0}^{n} a_\nu b^\nu$ $(n = 0, 1, 2, \ldots)$. Denote by p_n the largest prime divisor in the numerator of S_n.*

(i) *If $f(b) \neq 0$, then $\limsup_{n\to\infty} p_n = \infty$.*

(ii) *If $f(b)$ is an irrational number, then $\lim_{n\to\infty} p_n = \infty$. (This last statement implies that the limit exists in an extended sense.)*

By studying Popken's proof of this theorem, one sees that the condition that $f(z)$ is an algebraic function can be somewhat modified without affecting the proof. We illustrate this remark by using Popken's original proof, but with different hypotheses, to derive a similar result. This new version of Popken's theorem, combined with a result of Mahler [**3**, Theorem 1, p. 57], enables us to obtain an interesting consequence about transcendental values of lacunary analytic functions taken at rational points. This last result is an extension of a particular case of the following theorem due to Cohn [**2**].

COHN'S THEOREM. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^{e_k}$ be a lacunary power series with rational coefficients $a_k = p_k/q_k$ $(k \geq 0)$. Let R be the radius of convergence of f, $A_k = \max_{i=0,\ldots,k} |a_i|$, and M_k the least common multiple of $q_0, \ldots, q_k$. If*

$$\lim_{k\to\infty} \frac{e_k + \log M_k + \log A_k}{e_{k+1}} = 0,$$

then $f(b)$ is transcendental for every algebraic number b, with $0 < |b| < R$.

This formulation of Cohn's result is taken from Cijsouw and Tijdeman [**1**]; indeed, Cijsouw and Tijdeman generalized Cohn's result to the case where the a_k are algebraic integers.

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2. A new version of Popken's Theorem.

THEOREM 1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series which is not a polynomial with rational integral coefficients* a_n $(n = 0, 1, 2, \ldots)$ *converging in a neighbourhood* $|z| < R$ *of the origin. Let* b *denote a rational number such that* $0 < |b| < R$. *Let* $S_n = \sum_{\nu=0}^{n} a_\nu b^\nu$ $(n = 0, 1. 2, \ldots)$. *Let* p_n *denote the largest prime divisor in the numerator of* S_n.

(i) *If* $f(b)$ *is a nonzero algebraic number, then* $\limsup_{n\to\infty} p_n = \infty$.

(ii) *If* $f(b)$ *is an algebraic irrational number, then* $\lim_{n\to\infty} p_n = \infty$.

PROOF. (i) Put $b = u/v$, where u and $v > 0$ are rational integers. Then

$$S_n = \sum_{\nu=0}^{n} \frac{a_\nu u^\nu}{v^\nu} = \frac{x_n}{y_n},$$

with $y_n = v^n$ and x_n an integer for $n \geq 1$. Denote the prime divisors of v by $p_1, p_2, \ldots, p_g$. Now suppose the assertion $\limsup_{n\to\infty} p_n = \infty$ is false. Then all integers y_i, x_i have a finite number of prime divisors $p_1, p_2, \ldots, p_w$ $(w \geq g)$. Thus

$$x_i = \pm p_1^{\xi_1} p_2^{\xi_2} \cdots p_w^{\xi_w}, \qquad y_i = p_1^{\eta_1} p_2^{\eta_2} \cdots p_2^{\eta_w} \qquad (i = 0, 1, 2, \ldots),$$

where the ξ's and η's are nonnegative integers.

Since $0 < |b| < R$, there exists a positive number δ so small that $\omega := (\delta + 1/R)|b| < 1$. If R' $(\geq R)$ denotes the radius of convergence of $\sum a_n z^n$, then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R'}.$$

Hence, for sufficiently large i,

$$\left| f(b) - \frac{x_i}{y_i} \right| = \left| \sum_{\nu=i+1}^{\infty} a_\nu b^\nu \right| \leq \sum_{\nu=i+1}^{\infty} \left(\delta + \frac{1}{R} \right)^\nu |b|^\nu = \frac{\omega^{1+i}}{1-\omega}.$$

Choose a positive number k so small that $1/v^k > \omega$. Then for sufficiently large i,

$$y_i^{-k} = v^{-ki} > \omega^i/(1-\omega).$$

Hence, for sufficiently large i,

$$|f(b) - x_i/y_i| < \omega^i/(1-\omega) < y_i^{-k}.$$

Since $f(b) \neq 0$ is an algebraic number, by Ridout's theorem [**5**],

$$f(b) = x_i/y_i \quad \text{for sufficiently large } i.$$

It follows that $a_i b^i = S_i - S_{i-1} = 0$. Hence, $a_i = 0$ for sufficiently large i, so $f(z)$ is necessarily a polynomial, which is a contradiction.

(ii) The proof is similar to (i). We assume the assertion is false. Then there exists an increasing sequence (n_j) such that all numerators of S_{n_j} can be formed from a finite number of primes. By the same arguments as in (i), using (n_j) instead of (n) and S_{n_j} rather than S_n, we arrive at the fact that, for sufficiently large j, $f(b) = x_{n_j}/y_{n_j}$. This contradicts the irrationality of $f(b)$, and Theorem 1 is proved.

We remark that, with only a slight change in the proof, Theorem 1 is still true if the coefficients a_k satisfy the Eisenstein condition, i.e. $\exists N \in \mathbf{N}$ such that $N^k a_k \in \mathbf{Z}$ $(k = 0, 1, 2, \ldots)$.

3. Transcendental values of lacunary power series. For our application we need the following result of Mahler [**3**, Theorem 1, p. 57].

MAHLER'S THEOREM. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with rational integral coefficients converging in a neighbourhood $|z| < R$ of the origin. Suppose $f(z)$ satisfies the gap condition, that is, there are two infinite sequences of integers, $\{r_n\}$ and $\{s_n\}$, satisfying*

$$0 = s_0 \leq r_1 < s_1 \leq r_2 < s_2 \leq r_3 < s_3 \leq \cdots, \qquad \lim_{n\to\infty} \frac{s_n}{r_n} = \infty,$$

such that $a_h = 0$ if $r_n < h < s_n$, but $a_{r_n} \neq 0$, $a_{s_n} \neq 0$ $(n = 1, 2, 3, \ldots)$. Let b be an algebraic number satisfying $|b| < R$. Then $f(b)$ is algebraic if and only if there exists a positive integer $N = N(b)$ such that $P_n(b) = 0$ for all $n \geq N$, where

$$P_n(z) = \sum_{h=s_n}^{r_{n+1}} a_h z^h \qquad (n = 0, 1, 2, \ldots).$$

We are now ready to prove

THEOREM 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a lacunary power series with properties as in Mahler's Theorem. If b is a rational number satisfying $0 < |b| < R$, then $f(b)$ is either a rational or a transcendental number.*

PROOF. Suppose, to the contrary, that $f(b)$ is an algebraic irrational number. Then by Mahler's Theorem there exist a positive integer $N = N(b)$ and an increasing sequence of positive integers (n_j) such that $S_{n_j} = S_{r_N}$ for all j. Thus the limiting value of the largest prime divisors p_n of S_n (as n tends to infinity) either does not exist, or, if it does, it is never infinite. This contradicts Theorem 1(ii), and our result follows.

We conclude with a few remarks.

(1) The case where $s_n = r_{n+1}$ $(n = 0, 1, 2, \ldots)$ in Theorem 2 corresponds to a special case of Cohn's Theorem mentioned earlier. In this case $f(b)$ is necessarily a transcendental number, because, if not, Mahler's Theorem then implies $P_n(b) = a_{s_n} b^{s_n} = 0$ for all sufficiently large n, so $f(z)$ reduces to a polynomial, which is a contradiction.

(2) The possibility, in Theorem 2, of $f(b)$ being a rational number does indeed exist, as shown by the following example. Take for $n \geq 1$,

$$r_{n+1} = s_n + 1, \qquad a_{r_{n+1}}/a_{s_n} = a_{s_n+1}/a_{s_n} = -1/b.$$

Then $P_n(b) = 0$ for all $n \geq 1$, so $f(b) = a_0 + a_{r_1} b^{r_1}$ is a rational number.

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