INTEGRABILITY OF POWER SERIES

BABU RAM

ABSTRACT. The object of this paper is to obtain generalization of certain results of
Jain [3] and Woyczynski [5] concerning integrability of power series.

1. Introduction. A nondecreasing continuous real-valued function Φ defined on the nonnegative half line and vanishing only at the origin will be called an *Orlicz function* (OF). A function $\Phi \in$ OF is said to satisfy Δ_α ($\alpha > 0$) condition for large u if there are constants $c > 0$ and $u_0 \geqslant 0$ such that $\Phi(\alpha u) \leqslant c\Phi(u)$, $u \geqslant u_0$. A convex Orlicz function Φ satisfying the conditions $\lim_{u \to 0}(\Phi(u)/u) = 0$ and $\lim_{u \to \infty}(\Phi(u)/u) = \infty$ is called a *Young function* (YF). Function Φ belongs to YF if and only if it admits a representation $\Phi(u) = \int_0^u \phi(t)\,dt$, where $\phi(t)$, $t \geqslant 0$, is positive, vanishing only at the origin, continuous on the right, nondecreasing and $\lim_{t \to \infty} \phi(t) = \infty$. Then $\Phi(u)/u \leqslant \phi(u) < \Phi(2u)/u$. Let $\mathcal{M}$ be the class of Orlicz functions which satisfy the following condition:

($\mathcal{M}$) There exists a convex function Λ, $\lambda > 1$, and $0 < \alpha < 1$, such that the inequality $\Lambda(u) \leqslant \Phi^\alpha(u) \leqslant \lambda\Lambda(u)$ holds for all u.

Let $L_\Phi(X, \mu)$, where $\Phi \in \Delta_\alpha$, be the Orlicz space, that is, the set of all complex-valued measurable functions f on a measure space (X, μ) such that the modular $\int_X \Phi(|f(x)|)\,d\mu$ is finite. By Hardy-Orlicz space H_Φ we mean a closed subspace of $L_\Phi(\langle 0, 2\pi \rangle, dx)$ spanned over trigonometric polynomials of the form $f(t) = \sum_{n=0}^N a_n e^{int}$.

A sequence $\langle a_n \rangle$ of nonnegative numbers is said to be quasi monotone if for some $\alpha > 0$, $a_{n+1} \leqslant a_n(1 + \alpha/n)$ or equivalently $n^{-\beta}a_n \downarrow 0$ for some $\beta > 0$.

Let $\Psi(x)$ be a nondecreasing positive function such that $\Psi(x)/x^{1-\delta}$ decreases as x increases in $(0, a)$, where $a > 0$ and δ is a small constant depending on Ψ. Then $\Psi'(x)$ exists [2] almost everywhere and $\Psi'(x) \geqslant 0$.

2. Results. We prove the following:

THEOREM. *Let* $\Phi \in \Delta_\alpha \cap \mathcal{M} \cap$ YF *and* $f(x) = \sum_{k=0}^\infty a_k x^k$, $0 \leqslant x < 1$. *If* $\langle a_n \rangle$ *is a quasi monotone sequence such that*

$$(2.0) \qquad \sum_{k=n}^\infty |a_k - a_{k+1}| \leqslant Ka_n,$$

then the following four statements are equivalent:

(2.1) $$\frac{\Phi(|f(x)|)}{\Psi(1-x)} \in L(0,1);$$

(2.2) $$\frac{\Phi(|f(e^{ix})|)}{\Psi(x)} \in L(0,\pi);$$

(2.3) $$\sum_{n=1}^{\infty} \frac{\Phi(na_n)}{n^2\Psi(1/n)} < \infty;$$

(2.4) $$\sum_{n=1}^{\infty} \frac{\Phi(A_n)}{n^2\Psi(1/n)} < \infty.$$

In the special case of decreasing sequence $\langle a_n \rangle$ and $\Psi \equiv 1$ the above result was obtained by Woyczynski [5] and in the case $\Psi(x) = x^{-\gamma}$, $0 \leqslant \gamma < 1$, by Jain [3]. It may also be mentioned here that the author [4] also obtained some results of this type for an increasing, positive and convex function Φ with no restriction on a_n.

3. Lemmas. The proof of our theorem is based upon the following lemmas, which are generalizations of the lemmas of Woyczynski [5]. The proof of Lemma 1 also generalizes a classical Hardy's inequality (see [6, 9.16]). Let us write $F(x) = \int_0^x f(t)\, dt$.

LEMMA 1. *Let $\Phi \in \mathcal{M}$, $a \geqslant 0$. Then*

$$\int_0^a \frac{\Phi(F(x)/x)}{\Psi(x)}\, dx \leqslant K(\alpha) \int_0^a \frac{\Phi(f(x))}{\Psi(x)}\, dx.$$

PROOF. By virtue of the assumption and of the Jensen's inequality there exists an α, $0 < \alpha < 1$, such that

$$\Phi^{\alpha}(F(x)/x) \leqslant \lambda\left(\int_0^x \Phi^{\alpha}(f(t))x^{-1}\, dt\right),$$

whence

(3.1) $$\int_0^a \frac{\Phi(F(x)/x)}{\Psi(x)}\, dx \leqslant \lambda^{1/\alpha} \int_0^a \frac{\left[\int_0^x \Phi^{\alpha}(f(t))x^{-1}\, dt\right]^{1/\alpha}}{\Psi(x)}\, dx.$$

We consider the integral on the right-hand side of (3.1). We write

$$H(x) = \int_0^x \Phi^{\alpha}(f(t))\, dt.$$

Then the integral under consideration becomes

$$\int_0^a \frac{(H(x)/x)^{1/\alpha}}{\Psi(x)}\, dx.$$

Integration by parts yields

$$\int_0^a \frac{(H(x)/x)^{1/\alpha}}{\Psi(x)} dx = \frac{a(H(a)/a)^{1/\alpha}}{\Psi(a)} - \int_0^a \frac{(H(x)/x)^{1/\alpha-1}\Phi^\alpha(f(x))}{\alpha\Psi(x)} dx$$

$$+ \int_0^a \frac{(H(x)/x)^{1/\alpha}}{\alpha\Psi(x)} dx + \int_0^a \frac{(H(x))^{1/\alpha}\Psi'(x)}{x^{1/\alpha-1}(\Psi(x))^2} dx.$$

Thus, by the above, and by Hölder's inequality
(3.2)

$$\alpha\left(\frac{1}{\alpha} - 1\right)\int_0^a \frac{(H(x)/x)^{1/\alpha}}{\Psi(x)} dx \leqslant \int_0^a \frac{(H(x)/x)^{1/\alpha-1}\Phi^\alpha(f(x))}{\Psi(x)} dx$$

$$\leqslant \left(\int_0^a \frac{\Phi(f(x))}{\Psi(x)} dx\right)^\alpha \left(\int_0^a \frac{(H(x))^{1/\alpha}x^{-1/\alpha}}{\Psi(x)} dx\right)^{1-\alpha}.$$

Dividing both sides by the last factor on the right, we get

$$(1 - \alpha)\left(\int_0^a \frac{(H(x)/x)^{1/\alpha}}{\Psi(x)} dx\right)^\alpha \leqslant \left(\int_0^a \frac{\Phi(f(x))}{\Psi(x)} dx\right)^\alpha$$

whence

$$(3.3) \qquad \int_0^a \frac{(H(x)/x)^{1/\alpha}}{\Psi(x)} dx \leqslant K(\alpha)\int_0^a \frac{\Phi(f(x))}{\Psi(x)} dx.$$

The inequalities (3.1) and (3.3) then yield

$$\int_0^a \frac{\Phi(F(x)/x)}{\Psi(x)} dx \leqslant K(\alpha)\int_0^a \frac{\Phi(f(x))}{\Psi(x)} dx.$$

LEMMA 2. *Let* $\Phi \in \Delta_\alpha \cap \mathrm{YF}$ *and* $a \geqslant 0$. *If* $f(x)$ *is a nonnegative function, then*

$$\int_0^a \frac{\Phi(F(x))}{\Psi(x)} dx \leqslant K\int_0^a \frac{\Phi(xf(x))}{\Psi(x)} dx.$$

PROOF. Integration by parts and the given hypothesis yield

$$\int_0^a \frac{\Phi(F(x))}{\Psi(x)} dx = \frac{a\Phi(F(a))}{\Psi(a)} - \int_0^a \frac{x\phi(F(x))f(x)}{\Psi(x)} dx$$

$$\geqslant - \int_0^a \frac{xf(x)\phi(F(x))}{\Psi(x)} dx.$$

Now using Chen's inequality ([1], 3.10 for $t = 1$), we get on using $\Phi(u)/u \leqslant \phi(u) < \Phi(2u)/u$ and $\Phi(\alpha u) \leqslant c\Phi(u)$,

$$xf(x)\phi(F(x)) = \mathrm{Max}\{xf(x)\phi(xf(x)), F(x)\phi(F(x))\}$$

$$\leqslant \{\Phi(2xf(x)) + \Phi(2F(x))\} \leqslant c[\Phi(xf(x)) + \Phi(F(x))].$$

Therefore

$$\int_0^a \frac{\Phi(F(x))}{\Psi(x)} dx \geqslant -c\int_0^a \frac{\Phi(xf(x))}{\Psi(x)} dx - c\int_0^a \frac{\Phi(F(x))}{\Psi(x)} dx$$

whence

$$\int_0^a \frac{\Phi(F(x))}{\Psi(x)} dx \leqslant K\int_0^a \frac{\Phi(xf(x))}{\Psi(x)} dx.$$

4. Proof of the Theorem. We shall prove the following implications:(2.1) $\Leftrightarrow$ (2.4); (2.4) $\Rightarrow$ (2.2); (2.2) $\Rightarrow$ (2.3) and (2.3) $\Rightarrow$ (2.4).

(2.1) $\Rightarrow$ (2.4). In fact, if we write $1 - x = y$, then we have for $1/(n + 1) \leqslant y \leqslant 1/n$, $n \geqslant 2$,

$$f(1 - y) \geqslant \sum_{k=0}^n a_k(1 - y)^k \geqslant (1 - 1/n)^n \sum_{k=0}^n a_k \geqslant \frac{A_n}{4}.$$

Thus

$$f(1 - y) \geqslant KA_n \quad \text{for } 1/(n + 1) \leqslant y \leqslant 1/n \qquad (n = 2, 3, \ldots).$$

Now

$$\sum_{n=1}^\infty \frac{\Phi(A_n)}{n^2\Psi(1/n)} \leqslant K \sum_{n=1}^\infty \int_n^{n+1} \frac{t^{-2}\Phi(A_{[t]})}{\Psi(1/t)} dt$$

$$= K \sum_{n=1}^\infty \int_{1/(n+1)}^{1/n} \frac{\Phi(A_{[1/u]})}{\Psi(u)} du$$

$$= K \int_{1/2}^1 \frac{\Phi(A_{[1/u]})}{\Psi(u)} du + K \sum_{n=2}^\infty \int_{1/(n+1)}^{1/n} \frac{\Phi(A_{[1/u]})}{\Psi(u)} du$$

$$\leqslant K + K \int_0^{1/2} \frac{\Phi(f(1 - u))}{\Psi(u)} du$$

$$\leqslant K + K \int_0^1 \frac{\Phi(f(x))}{\Psi(1 - x)} dx < \infty.$$

(2.4) $\Rightarrow$ (2.1). We have

$$\int_0^1 \frac{\Phi(f(x))}{\Psi(1 - x)} dx = \sum_{n=2}^\infty \int_{1-1/(n+1)}^{1-1/n} \frac{\Phi(f(x))}{\Psi(1 - x)} dx$$

$$= \sum_{n=2}^\infty \int_{1/n}^{1/(n-1)} \frac{\Phi(f(1 - x))}{\Psi(x)} dx$$

$$= \sum_{n=2}^\infty \int_{1/n}^{1/(n-1)} \frac{\Phi\left(\sum_{k=0}^\infty a_k(1 - x)^k\right)}{\Psi(x)} dx$$

$$\leq \sum_{n=2}^{\infty} \int_{1/n}^{1/(n-1)} \frac{\Phi\left(\sum_{k=0}^{\infty} a_k (1 - 1/n)^k\right)}{\Psi(x)} dx$$

$$\leq \sum_{n=2}^{\infty} \frac{\Phi\left(\sum_{k=0}^{\infty} \sum_{j=nk}^{n(k+1)} a_j (1 - 1/n)^j\right)}{n^2 \Psi(1/n)}$$

$$\leq K \sum_{n=2}^{\infty} \frac{\Phi\left(\sum_{k=0}^{\infty} e^{-k} \sum_{j=0}^{n(k+1)} a_j\right)}{n^2 \Psi(1/n)}$$

$$\leq K \sum_{n=2}^{\infty} \frac{\Phi\left(\sum_{k=0}^{\infty} e^{-k}(k+1) A_n\right)}{n^2 \Psi(1/n)}$$

$$\leq K \sum_{n=2}^{\infty} \frac{\Phi(A_n)}{n^2 \Psi(1/n)} < \infty.$$

(2.4) $\Rightarrow$ (2.2). We have

$$\left|\mathrm{Re}\, f(e^{ix})\right| = \left|\sum_{k=1}^{\infty} a_k \cos kx\right|$$

$$= \left|\sum_{k=1}^{n} a_k \cos kx + \sum_{k=n+1}^{\infty} a_k \cos kx\right|$$

$$= \left|\sum_{k=1}^{n} a_k \cos kx + \sum_{k=n}^{\infty} (a_k - a_{k+1}) D_k(x) - a_n D_n(x)\right|$$

$$\leq A_n + O(1/x) \sum_{k=n}^{\infty} |a_k - a_{k+1}| + O(x^{-1} a_n).$$

Then, on account of (2.0), it follows that

$$\int_0^{\pi} \frac{\Phi\left(|\mathrm{Re}\, f(e^{ix})|\right)}{\Psi(x)} dx = \sum_{n=1}^{\infty} \int_{\pi/(n+1)}^{\pi/n} \frac{\Phi\left(|\mathrm{Re}\, f(e^{ix})|\right)}{\Psi(x)} dx$$

$$= \sum_{n=1}^{\infty} \int_{\pi/(n+1)}^{\pi/n} \frac{\Phi\left[A_n + O(1/x)(a_n) + O(1/x)(a_n)\right]}{\Psi(x)} dx$$

$$\leq K \sum_{n=1}^{\infty} \frac{\Phi(A_n + na_n)}{n^2 \Psi(1/n)}.$$

Since $\langle n^{-\beta} a_n \rangle$ is monotonically decreasing,

$$A_n = \sum_{k=1}^{n} k^{-\beta} a_k k^{\beta} \geq n^{-\beta} a_n \sum_{k=1}^{n} k^{\beta} \geq K n^{-\beta} a_n n^{\beta+1} = K n a_n.$$

Therefore, we have

$$\int_0^{\pi} \frac{\Phi\left(|\mathrm{Re}\, f(e^{ix})|\right)}{\Psi(x)} dx \leq K \sum_{n=1}^{\infty} \frac{\Phi(A_n)}{n^2 \Psi(1/n)} < \infty.$$

It can be proved similarly that

$$\int_0^\pi \frac{\Phi\left(\left|\operatorname{Im} f(e^{ix})\right|\right)}{\Psi(x)}\,dx < \infty.$$

(2.2) $\Rightarrow$ (2.3). Let $r(t) = \operatorname{Re} f(e^{it})$. The condition (2.0) implies that $\langle a_n \rangle$ is a sequence of bounded variation. Hence the Fourier series of r converges for $x > 0$. We write

$$R(t) = \int_0^t r(x)\,dx \quad \text{and} \quad R_1(t) = \int_0^t R(x)\,dx.$$

Integrating the Fourier series of $r(t)$, we have

$$R_1(t) = \sum_{j=1}^\infty a_j (1 - \cos jt)\, j^{-2}$$

$$\geqslant \sum_{j=1}^n a_j j^{-2} (1 - \cos jt)$$

$$= 2 \sum_{j=1}^n j^{-2} a_j \frac{\sin^2 jt}{2}$$

$$\geqslant 2 \sum_{j=1}^n j^{-2} a_j \frac{4}{\pi^2} \frac{j^2 t^2}{4}, \qquad \pi/(n+1) \leqslant t \leqslant \pi/n$$

$$\geqslant Kt^2 \sum_{j=1}^n a_j \geqslant Kt^2 n a_n \quad \text{due to the monotonicity of } \langle n^{-\beta} a_n \rangle.$$

Now, by Lemma 1, we have

$$\sum_{n=1}^\infty \frac{\Phi(na_n)}{n^2 \Psi(1/n)} \leqslant \sum_{n=1}^\infty \int_{\pi/(n+1)}^{\pi/n} \frac{\Phi\left(x^{-2} R_1(x)\right)}{\Psi(x)}\,dx$$

$$= \int_0^\pi \frac{\Phi\left(x^{-2} R_1(x)\right)}{\Psi(x)}\,dx$$

$$\leqslant \int_0^\pi \frac{\Phi\left(x^{-2} \int_0^x |R(t)|\,dt\right)}{\Psi(x)}\,dx$$

$$\leqslant \int_0^\pi \frac{\Phi\left(x^{-1} \int_0^x \frac{|R(t)|}{t}\,dt\right)}{\Psi(x)}\,dx$$

$$\leqslant K \int_0^\pi \frac{\Phi\left(\frac{|R(x)|}{x}\right)}{\Psi(x)}\,dx$$

$$\leqslant K \int_0^\pi \frac{\Phi\left(|r(x)|\right)}{\Psi(x)}\,dx$$

$$\leqslant K \int_0^\pi \frac{\Phi\left(|f(e^{ix})|\right)}{\Psi(x)}\,dx < \infty.$$

(2.3) $\Rightarrow$ (2.4). Let $a(x)$ be a function equal to a_n if $n - 1 \leqslant x < n$, $n = 1, 2, \ldots$, and let $A(x) = \int_0^x a(t)\, dt$. The assumption

$$\sum_{n=1}^{\infty} \frac{\Phi(na_n)}{n^2 \Psi(1/n)} < \infty$$

implies that $\Phi(ta(t))/\Psi(t)$ is integrable on the positive half line. Then, by virtue of Lemma 2, $\int_0^a (\Phi(A(t))/\Psi(t))\, dt < \infty$. But this is equivalent to the convergence of the series $\sum_{n=1}^{\infty} \Phi(A_n)/n^2 \Psi(1/n)$. This completes the proof of the theorem.

REFERENCES

1. Y. M. Chen, *Some asymptotic properties of Fourier constants and integrability theorems*, Math. Z. **68** (1957), 227–244.

2. _____, *Some further asymptotic properties of Fourier constants*, Math. Z. **69** (1958), 105–120.

3. P. Jain, *On the integrability of power series*, Proc. Amer. Math. Soc. **42** (1974), 569–574.

4. B. Ram, *Integrability theorems for power series*, Colloq. Math. **31** (1974), 293–299.

5. W. A. Woyczynski, *Positive coefficient elements of Hardy-Orlicz spaces*, Colloq. Math. **21** (1970), 103–110.

6. A. Zygmund, *Trigonometric series*, Vol. I and II, Cambridge Univ. Press, 1968.

DEPARTMENT OF MATHEMATICS, M. D. UNIVERSITY, ROHTAK - 124001, INDIA