PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 2, February 1985

REDUCTIVE ALGEBRAS CONTAINING A DIRECT SUM OF THE UNILATERAL SHIFT AND A CERTAIN OTHER OPERATOR ARE SELFADJOINT

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ABSTRACT. We give a partial solution of the reductive algebra problem to prove that: a reductive algebra containing the direct sum of a unilateral shift of finite multiplicity and a finite-dimensional completely nonunitary contraction is a von Neumann algebra.

1. Introduction. An algebra $\mathcal{U}$ of operators on a Hilbert space is reductive if it is weakly closed, contains the identity operator, and has the property that Lat $\mathcal{U}$ = Lat $\mathcal{U}^*$. Von Neumann algebras are reductive; the reductive algebra problem (posed in [**5**]) is: Is every reductive algebra a von Neumann algebra? There are a number of partial solutions to this problem [**1**, **2**, **3**, **4**, **5**, **6**, Theorem 9.15 and **7**].

By presenting a new technique in this paper, we generalize the result of [**4**] to prove that a reductive algebra containing the direct sum of a unilateral shift of finite multiplicity and a finite-dimensional completely nonunitary contraction is a von Neumann algebra. A consequence is a new proof of Burnside's Theorem. We conclude by presenting an open question which arises from our work.

2. Throughout we let $S \in \mathcal{L}(\mathcal{H})$ denote a unilateral shift of finite multiplicity, and $T \in \mathcal{L}(\mathcal{K})$ a finite-dimensional completely nonunitary contraction.

THEOREM. *If $\mathcal{U}$ is a reductive algebra containing the operator $S \oplus T$ then $\mathcal{U}$ is a von Neumann algebra.*

The proof requires the following known Lemma.

LEMMA [**4**]. *If $\mathcal{U}$ is a reductive algebra, and if the span of the ranges of the finite rank operators in $\mathcal{U}$ is the entire space, then $\mathcal{U}$ is a von Neumann algebra.*

PROOF OF THE THEOREM. We show that the hypothesis of the Theorem implies the hypothesis of the Lemma. To this end, let $P(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i)$ be the characteristic polynomial of T. Since T is a completely nonunitary contraction, it follows that

Received by the editors March 8, 1984.
1980 *Mathematics Subject Classification*. Primary 47C05, 47C15; Secondary 46C10.
[1]This research was supported in part by Grant No. MCS 82-01607 from the National Science Foundation.

0002-9939/85 $1.00 + $.25 per page

$|\lambda_i| < 1$ for $i = 1, 2, \ldots, m$. Define the finite Blaschke product

$$B(\lambda) = \prod_{i=1}^{m} (\lambda - \lambda_i)(1 - \bar{\lambda}_i \lambda)^{-1}.$$

It is easy to show that the operator $U = B(S)$ is unitarily equivalent to a unilateral shift of multiplicity mn and

$$(1) \qquad B(S \oplus T) = B(S) \oplus B(T) = U \oplus 0 \in \mathscr{U}.$$

Define the algebra

$$\mathscr{A} = \left\{ X \in \mathscr{L}(\mathscr{H}) \colon \text{ There are operators } Y \in \mathscr{L}(\mathscr{H}, \mathscr{K}) \text{ and } Z \in \mathscr{L}(\mathscr{K}) \text{ such that } \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix} \in \mathscr{U} \right\}.$$

We prove that $\mathscr{A}_w$, the closure of $\mathscr{A}$ in the weak operator topology of $\mathscr{L}(\mathscr{H})$, is a reductive algebra. To this end, let X be an operator in $\mathscr{A}$, let f be a nonzero vector in $\mathscr{H}$, and let ε be a positive number. There exist operators Y and Z such that $\left[\begin{smallmatrix} X & 0 \\ Y & Z \end{smallmatrix}\right] \in \mathscr{U}$. From (1) we conclude that

$$(U \oplus 0) \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix} = UX \oplus 0 \in \mathscr{U}.$$

Since $\mathscr{U}$ is reductive, there exists an operator $A = (A_{ij}) \in \mathscr{U}$ such that

$$\|(A - (X^* U^* \oplus 0))(Uf, 0)\| < \varepsilon;$$

hence

$$(2) \qquad \|(A_{11} U - X^*) f\| < \varepsilon.$$

Since the operator $A_{11} U \in \mathscr{A}$, it is easy to conclude from (2) that $\operatorname{Lat} \mathscr{A} = \operatorname{Lat} \mathscr{A}^*$. Therefore, $\mathscr{A}_w$ is a reductive algebra.

Now $\mathscr{A}_w$ is a reductive algebra containing U, a unilateral shift of finite multiplicity; thus $\mathscr{A}_w$ is a von Neumann algebra by [**4**, Theorem 1]. This implies that $U^* \in \mathscr{A}_w$; hence, there exists a net $\{X_\alpha\}_\alpha \subset \mathscr{A}$ such that

$$(3) \qquad U^* = \lim_\alpha X_\alpha \quad \text{(weak operator topology)}.$$

From the definition of $\mathscr{A}$, it follows that there are operators Y_α and Z_α such that

$$\begin{bmatrix} X_\alpha & 0 \\ Y_\alpha & Z_\alpha \end{bmatrix} \in \mathscr{U} \qquad \text{for every } \alpha.$$

Thus we have

$$(4) \qquad (U \oplus 0) \begin{bmatrix} X_\alpha & 0 \\ Y_\alpha & Z_\alpha \end{bmatrix} = (UX_\alpha \oplus 0) \in \mathscr{U}$$

for every α. Now (3) and (4) together imply that

$$UU^* \oplus 0 = (1 - p) \oplus 0 \in \mathscr{U};$$

hence,

$$F_0 = P \oplus 1 \quad \text{and} \quad F_k = U^k P \oplus 0 \in \mathscr{U}, \qquad \text{for } k \geqslant 1,$$

where P is the orthogonal projection of $\mathscr{H}$ onto $\ker U^*$. It is easy to show that

$$(5) \qquad \bigvee_{k \geqslant 0} \operatorname{ran} F_k = \mathscr{H} \oplus \mathscr{K}.$$

Since the operators F_k, $k \geqslant 0$, are of finite rank, it follows from (5) that the span of the ranges of all finite rank operators in $\mathcal{U}$ is the entire space. Therefore, the Lemma implies that $\mathcal{U}$ is a von Neumann algebra, which was to be shown $\square$

The following classical result is an easy consequence of the Theorem.

COROLLARY (BURNSIDE). *If $\mathcal{K}$ is a finite-dimensional Hilbert space and $\mathcal{U}$ is a subalgebra of $\mathcal{L}(\mathcal{K})$ with no nontrivial subspace, then $\mathcal{U} = \mathcal{L}(\mathcal{K})$.*

PROOF. Since Lat $\mathcal{U} = \{\{0\}, \mathcal{K}\}$, it follows that $\mathcal{U}$ is a reductive algebra. Now let $S \in \mathcal{L}(\mathcal{H})$ be a unilateral shift of finite multiplicity, and define the reductive algebra $\mathcal{W} = \mathcal{L}(\mathcal{H}) \oplus \mathcal{U}$. Since $\mathcal{W}$ contains the operator $S \oplus 0$, $\mathcal{W}$ is a von Neumann algebra by the Theorem, which implies that $\mathcal{U}$ is a von Neumann algebra. It easily follows from this fact that $\mathcal{U} = \mathcal{U}''$, where $\mathcal{U}''$ is the double commutant of $\mathcal{U}$.

The fact that $\mathcal{U}$ has no nontrivial invariant subspace implies that $\mathcal{U}'$, the commutant of $\mathcal{U}$, consists of the scalar operators. Therefore,

$$\mathcal{U} = \mathcal{U}'' = (\mathcal{U}')' = \{\lambda 1\colon\ \lambda \in \mathbf{C}\}' = \mathcal{L}(\mathcal{K}).$$

Thus $\mathcal{U} = \mathcal{L}(\mathcal{K})$, which was to be shown. $\square$

REMARK. If the Theorem remains true without the restriction that $\mathcal{K}$ be finite dimensional, then every reductive algebra is a von Neumann algebra. That is, the reductive algebra problem has a solution.

A completely nonunitary contraction $T \in \mathcal{L}(\mathcal{H})$ is a C_0-operator if there exists an inner function $f \in H^\infty(\mathbf{D})$ such that $f(T) = 0$. T is an essentially unitary C_0-operator if the defect operators $(1 - T^*T)_{\frac{1}{2}}$ and $(1 - TT^*)^{\frac{1}{2}}$ are compact.

Question. If $S \in \mathcal{L}(\mathcal{H})$ is a unilateral shift of finite multiplicity, if $T \in \mathcal{L}(\mathcal{K})$ is an essentially unitary C_0-operator, and if $\mathcal{U}$ is a reductive algebra containing the operator $S \oplus T$, must $\mathcal{U}$ be a von Neumann algebra?

ACKNOWLEDGEMENT. The author wishes to express his gratitude to Professor Allen Shields for his valuable guidance.

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