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POSITIVELY CURVED KAEHLER SUBMANIFOLDS

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ABSTRACT. In this note we prove that if the holomorphic curvature of a compact Kaehler submanifold in the complex projective space is bigger than $\frac{1}{2}$, then it is totally geodesic.

Introduction. Let CP^m be the m-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $c = 1$. We will prove the following result.

THEOREM. *Let M^n be a compact Kaehler submanifold of complex dimension n immersed in CP^m. If every holomorphic sectional curvature of M^n is greater than $\frac{1}{2}$, then M^n is totally geodesic in CP^m.*

The above Theorem was conjectured by K. Ogiue [**1**]. Some partial solutions are collected in [**1**]. As the Frankel conjecture is true [**2**], we remark that under the hypothesis of the theorem, M^n is biholomorphic to CP^n. We use the notation of [**1**].

1. Preliminaries. Let M^n be a Kaehler submanifold of complex dimension n in the complex projective space CP^m. We denote by J and g the complex structure and the metric of constant holomorphic sectional curvature $c = 1$ on CP^m. Let $\overline{\nabla}$ and ∇ be the Riemannian connection of CP^m and M^n, respectively, σ the second fundamental form of the immersion, A and $\nabla^\perp$ the Weingarten endomorphism and the normal connection of M^n in CP^m, and $\nabla\sigma$ the covariant derivative of σ.

We define the second covariant derivative of σ by

$$(1.1)\quad (\nabla^2\sigma)(X,Y,Z,W) = \nabla_X^\perp((\nabla\sigma)(Y,Z,W)) - (\nabla\sigma)(\nabla_X Y, Z, W) - (\nabla\sigma)(Y, \nabla_X Z, W) - (\nabla\sigma)(Y, Z, \nabla_X W)$$

for any vector fields X, Y, Z, W tangent to M^n.

Let $\overline{R}$, R and $R^\perp$ denote the curvature tensors associated with $\overline{\nabla}$, ∇ and $\nabla^\perp$, respectively. Then we have

$$(1.2)\quad \overline{R}(\overline{X},\overline{Y})\overline{Z} = \tfrac{1}{4}\{g(\overline{Y},\overline{Z})\overline{X} - g(\overline{X},\overline{Z})\overline{Y} + g(J\overline{Y},\overline{Z})J\overline{X} - g(J\overline{X},\overline{Z})J\overline{Y} + 2g(\overline{X},J\overline{Y})J\overline{Z}\},$$

$$(1.3)\quad R(X,Y)Z = \overline{R}(X,Y)Z + A_{\sigma(Y,Z)}X - A_{\sigma(X,Z)}Y,$$

$$(1.4)\quad R^\perp(X,Y,\xi,\eta) = \overline{R}(X,Y,\xi,\eta) + g([A_\xi, A_\eta]X, Y)$$

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for vector fields $\overline{X}, \overline{Y}, \overline{Z}$ tangent to CP^m, X, Y, Z tangent to M^n and ξ, η normal to M^n.

Moreover, σ and $\nabla\sigma$ are symmetric, and

$$(1.5)\quad (\nabla^2\sigma)(X, Y, Z, W) - (\nabla^2\sigma)(Y, X, Z, W) = R^{\perp}(X, Y)\sigma(Z, W) \\ -\sigma(R(X, Y)Z, W) - \sigma(Z, R(X, Y)W).$$

We also consider the relations

$$(1.6)\qquad \sigma(JX, Y) = \sigma(X, JY) = J\sigma(X, Y),$$

$$(1.7)\qquad A_{J\xi} = JA_{\xi}, \qquad JA_{\xi} = -A_{\xi}J, \qquad \nabla_X^{\perp} J\xi = J\nabla_X^{\perp}\xi,$$

$$(1.8)\qquad (\nabla\sigma)(JX, Y, Z) = (\nabla\sigma)(X, JY, Z) \\ = (\nabla\sigma)(X, Y, JZ) = J(\nabla\sigma)(X, Y, Z).$$

Finally if u is a unit tangent vector to M^n, then the holomorphic sectional curvature H of M^n determined by u is given by

$$(1.9)\qquad H(u) = 1 - 2\|\sigma(u, u)\|^2.$$

2. Proof of Theorem. Let Π: $UM \to M$ and UM_p be the unit tangent bundle of M and its fiber over $p \in M$, respectively. We define a function f: $UM \to \mathbf{R}$ by $f(u) = \|\sigma(u, u)\|^2$ for any u in UM. From (1.9), the hypothesis $H > \frac{1}{2}$ is equivalent to $f < \frac{1}{4}$.

Fix v in UM_p, $p \in M$. For any u in UM_p, let $\gamma_u(t)$ be the geodesic in M given by the initial conditions $\gamma_u(0) = p$, $\gamma_u'(0) = u$. By parallel translating v along $\gamma_u(t)$, we obtain a vector field $V_u(t)$. Put $f_u(t) = f(V_u(t))$. By direct computation we obtain

$$(2.1)\qquad \frac{d}{dt} f_u(t) = 2g\big((\nabla\sigma)(\gamma_u', V_u, V_u), \sigma(V_u, V_u)\big)(t),$$

and

$$(2.2)\qquad \frac{d^2}{dt^2} f_u(0) = 2g\big((\nabla^2\sigma)(u, u, v, v), \sigma(v, v)\big) + 2\|(\nabla\sigma)(u, v, v)\|^2.$$

As $(\nabla^2\sigma)(Jv, Jv, v, v) = (\nabla^2\sigma)(Jv, v, Jv, v)$, from (1.5) we have

$$(2.3)\qquad g\big((\nabla^2\sigma)(Jv, Jv, v, v) - (\nabla^2\sigma)(v, Jv, Jv, v), \sigma(v, v)\big) \\ = R^{\perp}(Jv, v, \sigma(Jv, v), \sigma(v, v)) \\ -2g(\sigma(R(Jv, v)v, Jv), \sigma(v, v)).$$

From (1.2)–(1.4), (1.6), (1.7) and (2.3), and taking into account that $(\nabla^2\sigma)(v, Jv, Jv, v) = -(\nabla^2\sigma)(v, v, v, v)$, we obtain

$$(2.4)\qquad g\big((\nabla^2\sigma)(Jv, Jv, v, v), \sigma(v, v)\big) \\ = -g\big((\nabla^2\sigma)(v, v, v, v), \sigma(v, v)\big) \\ + \tfrac{3}{2}\|\sigma(v, v)\|^2 - 6g\big(\sigma(v, v), \sigma(A_{\sigma(v, v)}v, v)\big).$$

Hence from (1.8), (2.2) and (2.4) we have

$$\frac{d^2}{dt^2} f_v(0) + \frac{d^2}{dt^2} f_{Jv}(0) = 3\|\sigma(v, v)\|^2 - 12 g\big(\sigma(v, v), \sigma\big(A_{\sigma(v,v)}v, v\big)\big) + 4\|(\nabla\sigma)(v, v, v)\|^2. \tag{2.5}$$

UM being compact, f attains the maximum at some vector in UM. We suppose that this vector is v. Hence

$$\frac{d^2}{dt^2} f_v(0) + \frac{d^2}{dt^2} f_{Jv}(0) \leqslant 0. \tag{2.6}$$

For any u in UM_p, with $g(v, u) = 0$, let $\alpha(s)$ be a curve in the sphere UM_p such that $\alpha(0) = v$, $\alpha'(0) = u$. As v is a critical point of f we have

$$\frac{d}{ds}(f \circ \alpha)(0) = 4g(\sigma(v, v), \sigma(v, u)) = 0,$$

or equivalently $g(A_{\sigma(v,v)}v, u) = 0$ for all $u \in UM_p$ with $g(v, u) = 0$. Hence

$$A_{\sigma(v,v)}v = \|\sigma(v, v)\|^2 v. \tag{2.7}$$

From (2.5) and (2.7) we obtain

$$\frac{d^2}{dt^2} f_v(0) + \frac{d^2}{dt^2} f_{Jv}(0) = 3f(v)(1 - 4f(v)) + 4\|(\nabla\sigma)(v, v, v)\|^2. \tag{2.8}$$

Finally, from (2.6) and (2.8) we conclude that $f(v)(1 - 4f(v)) \leqslant 0$. From the hypothesis, $f(v) < \frac{1}{4}$. Hence $f(v) = 0$, M^n is totally geodesic, and we have proved the Theorem.

References

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